Proof Theory for Theories of Ordinals III:Π_N-Reflection

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July 5, 2010

Abstract

This paper deals with a proof theory for a theory T_N of Π_N-reflecting ordinals using a system $Od(\Pi_N)$ of ordinal diagrams. This is a sequel to the previous one [6] in which a theory for Π_3-reflecting ordinals is analysed proof-theoretically.

1 Prelude

This is a sequel to the previous ones [5] and [6]. Namely our aim here is to give finitary analyses of finite proof figures in a theory for Π_N-reflecting ordinals, [14] via cut-eliminations as in Gentzen-Takeuti's consistency proofs, [10] and [15]. Throughout this paper N denotes a positive integer such that $N \geq 4$.

Let T be a theory of ordinals. Let Ω denote the (individual constant corresponding to the) ordinal ω_1^{CK}. We say that T is a Π_2^Ω-*sound theory* if

$$\forall \Pi_2\, A(T \vdash A^\Omega \Rightarrow A^\Omega).$$

Definition 1.1 (Π_2^Ω-*ordinal of a theory*) Let T be a Π_2^Ω-sound and recursive theory of ordinals. For a sentence A let A^α denote the result of replacing unbounded quantifiers $Qx\,(Q \in \{\forall, \exists\})$ in A by $Qx < \alpha$. Define the Π_2^Ω-*ordinal* $|T|_{\Pi_2^\Omega}$ *of* T by

$$|T|_{\Pi_2^\Omega} := \inf\{\alpha \leq \omega_1^{CK} : \forall \Pi_2 \text{ sentence } A(T \vdash A^\Omega \Rightarrow A^\alpha)\} < \omega_1^{CK}$$

Roughly speaking, the aim of proof theory for theories T of ordinals is to describe the ordinal $|T|_{\Pi_2^\Omega}$. This gives Π_2^Ω-ordinal of an equivalent theory of sets, cf. [5].

Let $\mathrm{KP}\Pi_N$ denote the set theory for Π_N-reflecting universes. $\mathrm{KP}\Pi_N$ is obtained from the Kripke-Platek set theory with the Axiom of Infinity by adding the axiom: for any Π_N formula $A(u)$

$$A(u) \to \exists z(u \in z \,\&\, A^z(u)).$$

In [9] we introduced a recursive notation system $Od(\Pi_N)$ of ordinals, which we studied first in [1]. An element of the notation system is called an *ordinal diagram* (henceforth abbreviated by *o.d.*). The system is designed for proof theoretic study of theories of Π_N-reflection. We [9] showed that for each $\alpha < \Omega$ in $Od(\Pi_N)$ $\mathrm{KP}\Pi_N$ proves that the initial segment of $Od(\Pi_N)$ determined by α is a well ordering.

Let T_N denote a theory of Π_N-reflecting ordinals. The aim of this paper is to show an upper bound theorem for the ordinal $|\mathrm{T}_N|_{\Pi_2^\Omega}$:

Theorem 1.1 $\forall \Pi_2\, A(\mathrm{T}_N \vdash A^\Omega \Rightarrow \exists\alpha \in Od(\Pi_N) \mid \Omega\, A^\alpha)$

Combining Theorem1.1 with the result in [9] mentioned above yields the:

Theorem 1.2 $|\mathrm{KP}\Pi_N|^\Omega_{\Pi_2} = |\mathrm{T}_N|^\Omega_{\Pi_2} =$ the order type of $Od(\Pi_N) \mid \Omega$

Proof theoretic study for Π_N-reflecting ordinals via ordinal diagrams were first obtained in a handwritten note [2].

For an alternative approach to ordinal analyses of set theories, see M. Rathjen's papers [11], [12] and [13].

Let us mention the contents of this paper.

In Section 2 a preview of our proof-theoretic analysis for Π_N-reflection is given. As in [6] inference rules $(c)^\sigma_{\alpha_1}$ are added to analyse an inference rule (Π_N-rfl) saying the universe of the theory T_N is Π_N-reflecting. A *chain* is defined to be a consecutive sequence of rules (c).

In Subsection 2.1 we expound that chains have to merge each other for a proof theoretic analysis of T_N for $N \geq 4$. An ordinal diagram in the system $Od(\Pi_N)$ defined in [9] may have its Q part, which has to obey complicated requirements. In Subsection 2.2 we explain what parts correspond to the Q part in proof figures.

In Section 3 the theory T_N for Π_N-reflecting ordinals is defined. In Section 4 let us recall briefly the system $Od(\Pi_N)$ of ordinal diagrams (abbreviated by o.d.'s) in [9].

In Section 5 we extend T_N to a formal system T_{Nc}. The language is expanded so that individual constants c_α for o.d.'s $\alpha \in Od(\Pi_N) \mid \pi$ are included. Inference rules $(c)^\sigma_{\alpha_1}$ are added. *Proofs* in T_{Nc} defined in Definition 5.8 are proof figures enjoying some provisos and obtained from given proofs in T_N by operating rewriting steps. Some lemmata for proofs are established. These are needed to verify that rewrited proof figures enjoy these provisos. To each proof P in T_{Nc} an o.d. $o(P) \in Od(\Pi_N) \mid \Omega$ is attached. Then the Main Lemma 5.1 is stated as follows: If P is a proof in T_{Nc}, then the endsequent of P is true.

In Section 6 the Main Lemma 5.1 is shown by a transfinite induction on $o(P) \in Od(\Pi_N) \mid \Omega$.

This paper relies heavily on the previous ones [5] and [6].

General Coventions. Let $(X, <)$ be a quasiordering. Let F be a function $F : X \ni \alpha \mapsto F(\alpha) \subseteq X$. For subsets $Y, Z \subset X$ of X and elements $\alpha, \beta \in X$, put

1. $\alpha \leq \beta \Leftrightarrow \alpha < \beta$ or $\alpha = \beta$
2. $Y \mid \alpha = \{\beta \in Y : \beta < \alpha\}$
3. $Y < Z :\Leftrightarrow \exists \beta \in Z \forall \alpha \in Y (\alpha < \beta)$
4. $Y < \beta :\Leftrightarrow Y < \{\beta\} \Leftrightarrow \forall \alpha \in Y (\alpha < \beta)$; $\alpha < Z :\Leftrightarrow \{\alpha\} < Z$
5. $Z \leq Y :\Leftrightarrow \forall \beta \in Z \exists \alpha \in Y (\beta \leq \alpha)$
6. $\beta \leq Y :\Leftrightarrow \{\beta\} \leq Y \Leftrightarrow \exists \alpha \in Y (\beta \leq \alpha)$; $Z \leq \alpha :\Leftrightarrow Z \leq \{\alpha\}$
7. $F(Y) = \bigcup \{F(\alpha) : \alpha \in Y\}$

2 A preview of proof-theoretic analysis

In this section a preview of our proof-theoretic analysis for Π_N-reflection is given.

Let us recall briefly the system $Od(\Pi_N)$ of o.d.'s in [9]. The main constructor in $Od(\Pi_N)$ is to form an o.d. $d^q_\sigma \alpha$ from a symbol d and o.d.'s in $\{\sigma, \alpha\} \cup q$, where σ denotes a recursively Mahlo ordinal and $q = Q(d^q_\sigma \alpha)$ a finite sequence of quadruples of o.d.'s called Q *part* of $d^q_\sigma \alpha$. By definition we set $d^q_\sigma \alpha < \sigma$. Let $\gamma \prec_2 \delta$ denote the transitive closure of the relation $\{(\gamma, \delta) : \exists q, \alpha(\gamma = d^q_\delta \alpha)\}$, and $\preceq_2$ its reflexive closure. Then the set $\{\tau : \sigma \prec_2 \tau\}$ is finite and linearly ordered by $\prec_2$ for each σ.

An o.d. of the form $\rho = d^q_\sigma \alpha$ is introduced in proof figures only when an inference rule (Π_N-rfl) for Π_N-reflection is resolved by using an inference rule $(c)_\rho$.

q in $\rho = d^q_\sigma \alpha$ includes some data $st_i(\rho), rg_i(\rho)$ for $2 \leq i < N$. $st_{N-1}(\rho)$ is an o.d. less than $\varepsilon_{\pi+1}$ and $rg_{N-1}(\rho) = \pi$, while $st_i(\rho), rg_i(\rho)$ for $i < N-1$ may be undefined. If these are defined, then we write $rg_i(\rho) \downarrow$, etc. and $\kappa = rg_i(\rho)$ is an o.d. such that $\rho \prec_i \kappa$, where $\gamma \prec_i \delta$ is a transitive closure of the relation $pd_i(\gamma) = \delta$ on o.d.'s such that $\prec_{i+1} \subseteq \prec_i$ and $\prec_2$ is the same as one mentioned above. q also includes data $pd_i(\rho)$. $st_{N-1}(\rho)$ is defined so that

$$\gamma \prec_{N-1} \rho \Rightarrow st_{N-1}(\gamma) < st_{N-1}(\rho) \tag{1}$$

In Subsection 2.2 we explain what parts correspond to the Q part in proof figures.

A theory T_N for Π_N-reflection is formulated in Tait's logic calculus, i.e., one-sided sequent calculus and $\Gamma, \Delta \ldots$ denote a *sequent*, i.e., a finite set of formulae. T_N has the inference rule (Π_N-rfl):

$$\frac{\Gamma, A \quad \neg\exists z A^z, \Gamma}{\Gamma} \ (\Pi_N\text{-rfl})$$

where $A \equiv \forall x_N \exists x_{N-1} \cdots Q x_1 B$ with a bounded formula B.

So (Π_N-rfl) says $A \to \exists z A^z$. [1]

To deal with the inference rule (Π_N-rfl) we introduce new inference rules $(c)^\sigma_\rho$ and $(\Sigma_i)^\sigma$ $(1 \leq i \leq N)$ as in [6]:

$$\frac{\Gamma, \Lambda^\sigma}{\Gamma, \Lambda^\rho} \ (c)^\sigma_\rho$$

where Λ is a set of Π_N-sentences as above, $\Lambda^\sigma = \{A^\sigma : A \in \Lambda\}$, the side formulae Γ consists solely of Σ^σ_1-sentences and ρ is of the form $d^q_\sigma \alpha$.

$$\frac{\Gamma, \neg A^\sigma \quad A^\sigma, \Lambda}{\Gamma, \Lambda} \ (\Sigma_i)^\sigma$$

where A is a Σ_i sentence. Although this rule $(\Sigma_i)^\sigma$ is essentially a (cut) inference, we need to distinguish between this and (cut) to remember that a (Π_N-rfl) was resolved.

When we apply the rule $(c)^\sigma_\rho$ it must be the case:

$$\text{any instance term } \beta < \sigma \text{ for the existential quantifiers } \exists x_{N-i} < \sigma (i\text{:odd}) \text{ in } A^\sigma \equiv \forall x_N < \sigma \exists x_{N-1} < \sigma \cdots Q x_1 < \sigma B \text{ is less than } \rho \tag{2}$$

As in [6] an inference rule (Π_N-rfl) is resolved by forming a succession of rules (c)'s, called a *chain*, which grows downwards in proof figures. We have to pinpoint, for each (c), the unique chain, which describes how to introduce the (c). To retain the uniqueness of the chain, i.e., not to branch or split a chain, we have to be careful in resolving rules with two uppersequents. Our guiding principles are:

(ch1) For any $\dfrac{A^\sigma}{A^\tau} \ (c)^\sigma_\tau$ with $\tau = d^q_\sigma \alpha$, if an o.d. β is substituted for an existential quantifier $\exists y < \sigma$ in A^σ, i.e., β is a realization for $\exists y < \sigma$, then $\beta < \tau$, cf. (2),
and

(ch2) Resolving rules having several upperseuquents must not branch a chain.

[1] For simplicity we suppress the parameter. Correctly $\forall u(A(u) \to \exists z(u < z \,\&\, A^z(u)))$.

2.1 Merging chains

As contrasted with [6] for Π_3-reflection we have to merge chains here. Let us explain this phenomenon.

We omit side formulae in this subsection.

1) First resolve a (Π_N-rfl) in the left figure, and resolve the $(\Sigma_N)^\sigma\ J_0$ to the right figure with a $\Sigma_{N-1}\ A_1$:

$$\dfrac{\dfrac{A}{A^\sigma}\,(c)^\pi_\sigma \qquad \neg A^\sigma}{}\,(\Sigma_N)^\sigma\ J_0 \qquad\qquad \dfrac{\dfrac{\dfrac{A}{A^\sigma} \quad \neg A^\sigma, \neg A_1^\sigma}{\neg A_1^\sigma} \qquad \dfrac{A_1}{A_1^\sigma}\,(c)^\pi_\sigma\ I_0}{}\,J_1\ (\Sigma_{N-1})^\sigma$$

with $A \equiv \forall x_N \exists x_{N-1} \forall x_{N-2} A_3$, $\sigma = d^\alpha_\pi \alpha$, where $A_3 \equiv \exists x_{N-3} A_4$ is a Σ_{N-3}-formula and α denotes the o.d. attached to the uppersequent A of $(c)^\pi_\sigma$.

2) Second resolve a (Π_N-rfl) above the $(c)^\pi_\sigma\ I_0$ and a (Σ_N) as in 1):

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg A_1^\sigma \quad \dfrac{\overset{P_1}{A_1, B}}{A_1^\sigma, B^\sigma}\,(c)^\pi_\sigma\ \tilde{I}_0}{B^\sigma}\,(\Sigma_{N-1})^\sigma}{B^\tau} \qquad \dfrac{\neg A_1^\sigma \quad \dfrac{A_1, \neg B^\tau, \neg B_1^\tau}{A_1^\sigma, \neg B^\tau, \neg B_1^\tau}}{\neg B^\tau, \neg B_1^\tau}}{\neg B_1^\tau} \qquad \dfrac{\dfrac{\neg A_1^\sigma \quad \dfrac{A_1, B_1}{A_1^\sigma, B_1^\sigma}\,(c)^\pi_\sigma}{B_1^\sigma}\,J_1}{B_1^\tau}\,(c)^\sigma_\tau\ I_1}{}$$

Fig.1

with a $\tau = d^\nu_\sigma \beta$ and a $\Sigma_{N-1}\ B_1 \equiv \exists y_{N-1} \forall y_{N-2} B_3$, where ν denotes the o.d. attached to the subproof P_1 ending with the uppersequent A_1, B of $(c)^\pi_\sigma\ \tilde{I}_0$. After that resolve the $(\Sigma_{N-1})^\sigma\ J_1$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg A_1^\sigma \quad \dfrac{\overset{P_2}{A_1, B}}{A_1^\sigma, B^\sigma}\,\tilde{I}_0}{B^\sigma}}{B^\tau} \qquad \neg B^\tau, \neg B_1^\tau}{\neg B_1^\tau} \qquad \dfrac{\dfrac{\dfrac{\neg A_1^\sigma \quad \dfrac{A_1, B_1, A_2}{A_1^\sigma, B_1^\sigma, A_2^\sigma}}{B_1^\sigma, A_2^\sigma} \qquad \dfrac{\dfrac{A}{A^\sigma} \quad \neg A^\sigma, \neg A_2^\sigma}{\neg A_2^\sigma}\,J_0'}{B_1^\sigma}}{B_1^\tau}}{}$$

Then resolve the $(\Sigma_N)^\sigma\ J_0'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg A_1^\sigma \quad \dfrac{\overset{P_3}{A_1, B}}{A_1^\sigma, B^\sigma}\,\tilde{I}_0}{B^\sigma}}{B^\tau} \qquad \neg B^\tau, \neg B_1^\tau}{\neg B_1^\tau} \qquad \dfrac{\dfrac{B_1^\sigma, A_2^\sigma \qquad \dfrac{\neg A_2^\sigma, \neg \tilde{A}_1^\sigma \quad \dfrac{\overset{\overset{(\Pi_N\text{-rfl})\ H}{\vdots}}{\tilde{A}_1}}{\tilde{A}_1^\sigma}\,(c)^\pi_\sigma\ I_0'}{\neg A_2^\sigma}\,(\Sigma_{N-2})^\sigma\ J_2}{B_1^\sigma}}{B_1^\tau}\,(c)^\sigma_\tau\ I_1}{}\,K$$

3) Thirdly resolve a (Π_N-rfl) H above the $(c)^\pi_\sigma\, I_0'$. One cannot resolve the (Π_N-rfl) H by introducing a $(c)^\sigma_\rho$ with $\rho < \tau$. Let me explain the reason.

Suppose that we introduce a new $(c)^\sigma_\rho\, I_1'$ with $\rho = d^\eta_\sigma \gamma$ immediately above the $(\Sigma_{N-2})^\sigma\, J_2$ as in [6]. Then the new $(c)^\sigma_\rho\, I_1'$ is introduced after the $(c)^\sigma_\tau\, I_1$ and so $\rho = d^\eta_\sigma \gamma < \tau$. Hence a new $(\Sigma_N)^\rho\, K'$ is introduced below the $(\Sigma_{N-1})^\tau\, K$:

$$\dfrac{\dfrac{\neg B_1^\tau \qquad \dfrac{\dfrac{B_1^\sigma, A_2^\sigma \qquad \dfrac{\dfrac{\neg A_2^\sigma, \neg\tilde{A}_1^\sigma \qquad \dfrac{\tilde{A}_1, D}{\tilde{A}_1^\sigma, D^\sigma}\,(c)^\pi_\sigma}{\neg A_2^\sigma, D^\sigma}}{\neg A_2^\sigma, D^\rho}\,(c)^\sigma_\rho\, I_1'}{B_1^\sigma, D^\rho}\,J_2}{B_1^\tau, D^\rho}\,(c)^\sigma_\tau\, I_1}{D^\rho}\,K \qquad\qquad \neg D^\rho}{}\,K' \qquad Fig.2$$

with $D \equiv \forall z_N \exists z_{N-1} \forall z_{N-2} D_3$. Nevertheless this does not work, because $\neg A_2 \equiv \exists x_{N-3} \forall x_{N-4} \neg A_4$ is a Σ_{N-2} sentence with $N-2 \geq 2$. Namely the principle **(ch1)** may break down for the $(c)^\sigma_\rho\, I_1'$ since any o.d. $\delta < \sigma$, i.e., possibly $\delta \geq \rho$ may be an instance term for the existential quantifier $\exists x_{N-3}$ in $A_2 \equiv \forall x_{N-2} \exists x_{N-3} A_4$ and may be substituted for the variable x_{N-3} in $\neg A_2^\sigma$. Only we knows that such a δ is less than σ and comes from the left upper part of J_2.

4) Therefore the chain for H has to connect or merge with the chain $I_0 - I_1$ for B:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A_1^\sigma \qquad \dfrac{\genfrac{}{}{0pt}{}{P_4}{A_1, B}}{A_1^\sigma, B^\sigma}\,\tilde{I}_0}{B^\sigma}}{B^\tau}\,I_1' \qquad \neg B^\tau, \neg B_1^\tau}{\neg B_1^\tau} \qquad \dfrac{\dfrac{B_1^\sigma, A_2^\sigma \qquad \dfrac{\dfrac{\dfrac{A}{A^\sigma}\,I_0'' \qquad \neg A^\sigma, \neg A_2^\sigma, \neg\tilde{A}_1^\sigma}{\neg A_2^\sigma, \neg\tilde{A}_1^\sigma}\,J_0' \qquad \dfrac{\tilde{A}_1, D}{\tilde{A}_1^\sigma, D^\sigma}\,I_0'}{\neg A_2^\sigma, D^\sigma}}{B_1^\sigma, D^\sigma}\,J_2}{B_1^\tau, D^\tau}\,I_1}{D^\tau}}{D^\rho}\,(c)^\tau_\rho\, I_2$$

$Fig.3$

with $\rho = d^\eta_\tau \gamma$ and a $(\Sigma_N)^\rho$ with the cut formula D^ρ follows this figure as in $Fig.2$, where η denotes the o.d. attached to the uppersequent $\tilde{A}_1, D$ of $(c)^\pi_\sigma\, I_0'$. $(\Sigma_{N-2})^\sigma\, J_2$ is a merging point for chains $I_0 - I_1$ and $I_0' - I_1 - I_2$.

The principle **(ch1)** for the new $(c)^\tau_\rho\, I_2$ will be retained for the simplest case $N = 4$ as in [6]. The problem is that the proviso (1) may break down: it may be the case $\nu = st_{N-1}(\tau) \leq st_{N-1}(\rho) = \eta$ since we cannot expect the upper part of $(c)^\pi_\sigma\, I_0'$ is simpler than the one of $(c)^\pi_\sigma\, \tilde{I}_0$.

In other words a new succession $I_0' - I_1 - I_2$ of collapsings starts. This is required to resolve Σ^σ_{N-2} sentence $\neg A_2^\sigma$ $(N - 2 \geq 2)$ and hence σ has to be Π_{N-1}-reflecting.

If this chain $I_0' - I_1 - I_2$ would grow downwards as in Π_3-reflection, i.e., in a chain $I_0' - I_1 - I_2 - \cdots - I_n$, I_n would come only from the upper part of I_0', then the proviso (1) would suffice to kill this process. But the whole process may be iterated : in $Fig.3$ another succession $I_0'' - I_1 - I_2 - I_3$ may arise by resolving the $(\Sigma_N)^\sigma\, J_0'$.

Nevertheless still we can find a reducing part, that is, the upper part of the $(c)^\tau_\rho\, I_2$: the upper part of the $(c)^\tau_\rho\, I_2$ becomes simpler in the step $I_2 - I_3$. Furthermore in the general case $N > 4$ merging processes could be iterated, vz. the merging point $(\Sigma_{N-2})^\sigma\, J_2$ may be resolved into a $(\Sigma_{N-3})^{\rho_1}$, which becomes a new merging point to analyse a Σ_{N-3} sentence $A_3^{\rho_1}$ where $\rho_1 \preceq \rho$ is a Π_{N-2}-reflecting and so on. Therefore in $Od(\Pi_N)$ the Q part of an o.d. may consist of several factors:

$$(\tau, \alpha, q = \{\nu_i, \kappa_i, \tau_i : i \in In(\rho)\}) \mapsto d^q_\tau \alpha = \rho$$

with $\kappa_{N-1} = rg_{N-1}(\rho) = \pi$. $In(\rho)$ denotes a set such that

$$N - 1 \in In(\rho) \subseteq \{i : 2 \leq i \leq N - 1\}.$$

We set for $i \in In(\rho)$:

$$st_i(\rho) = \nu_i, rg_i(\rho) = \kappa_i, pd_i(\rho) = \tau_i.$$

If $i \notin In(\rho)$, set

$$pd_i(\rho) = pd_{i+1}(\rho), st_i(\rho) \simeq st_i(pd_i(\rho)), rg_i(\rho) \simeq rg_i(pd_i(\rho)).$$

Also these are defined so that $pd_2(\rho) = \tau$ for $\rho = d^q_\tau \alpha$.

For the o.d. $\rho = d^q_\tau \gamma$ in the $Fig.3$, $In(\rho) = \{N-2, N-1\}, st_{N-1}(\rho) = \eta, pd_{N-1}(\rho) = \sigma, rg_{N-2}(\rho) = \tau = pd_{N-2}(\rho), st_{N-2}(\rho) = \gamma = st_2(\gamma)$.

Thus $\nu_i = st_i(\rho)$ corresponds to the upper part of a $(c)^{rg_i(\rho)}$ while $\tau_{N-1} = pd_{N-1}(\rho)$ indicates that the first, i.e., uppermost merging point for a chain ending with a $(c)_\rho$ is a rule $(\Sigma_{N-2})^{\tau_{N-1}}$, e.g., the rule J_2 in $Fig.3$. Note that $st_{N-1}(\rho) = \eta < st_{N-1}(pd_{N-1}(\rho))$, cf. (1). $\kappa_i = rg_i(\rho)$ is an o.d. such that there exists a $(c)^{\kappa_i}$ in the chain for $(c)_\rho$. We will explain how to determine the rule $(c)^{rg_i(\rho)}$, i.e., the point to which we direct our attention in Subsection 2.2.

The case $In(\rho) = \{N-1\}$ corresponds to the case when a $(c)^{pd_{N-1}(\rho)}_\rho$ is introduced without merging points, i.e., as a resolvent of a (Π_N-rfl) above the top of the chain whose bottom is a $(c)_{pd_{N-1}(\rho)}$. The case $In(\rho) = \{N-2, N-1\}$ corresponds to the case when a $(c)^{pd_2(\rho)}_\rho$ $(pd_2(\rho) = pd_{N-2}(\rho))$ is introduced with a merging point $(c)^{pd_{N-1}(\rho)}$.

In $Fig.3$ a new succession with a merging point $(c)^\tau_\rho\, I_2$ arises by resolving a $(\Sigma_N)^\tau$ below the $(c)^\sigma_\tau\, I_1'$, i.e., $\tilde{I}_0 - I_1' - I_2 - I_3$ $(c)^\rho_\kappa$ for a κ with a $\lambda = st_{N-1}(\kappa)$. But in this case we have

$$\lambda = st_{N-1}(\kappa) < st_{N-1}(\tau) = \nu.$$

$st_{N-1}(\kappa)$ corresponds to the upper part P_1 of a $(c)^\pi_\sigma\, \tilde{I}_0$ in $Fig.1$, when the $(c)^\sigma_\tau$ was originally introduced. This part P_1 is unchanged up to $Fig.3$:

$P_1 = P_2 = P_3 = P_4$. Roughly speaking, $\tilde{I}_0 - I'_1 - I_3$ can be regarded as a Π_{N-1} resolving series $I_0 - I_1 - I_3$. This prevents the new merging points from going downwards unlimitedly.

2.2 The Q part of an ordinal diagram

In this subsetion we explain how to determine the Q part q of $\rho = d^q_\sigma \alpha$ from a proof figure when an inference rule $(c)^\sigma_\rho$ is introduced.

In general such a $(c)^\sigma_\rho$ is formed when we resolve an inference rule $(\Pi_N\text{-rfl})\, H$:

$$
\begin{array}{cc}
 & (\Pi_N\text{-rfl})\, H \\
 & \vdots \\
 & \dfrac{\Gamma_0}{\Gamma'_0}\ (c)^{\pi}_{\sigma_1}\ J_0 \\
\vdots\ \mathcal{C}_{n_{m+1}} & \vdots \\
\dfrac{\Gamma^{m+1}_p}{(\Gamma'_p)^{m+1}}\ (c)^{\sigma_p}_{\sigma_{p+1}}\ J^{m+1}_p & \dfrac{\Gamma_p}{\Gamma'_p}\ (c)^{\sigma_p}_{\sigma_{p+1}}\ J_p \\
\vdots & \vdots \\
\dfrac{\Gamma^{m+1}_{n_m}}{(\Gamma'_{n_m})^{m+1}}\ (c)^{\sigma_{n_m}}_{\sigma_{n_m+1}}\ J^{m+1}_{n_m} & \dfrac{\Gamma_{n_m}}{\Gamma'_{n_m}}\ (c)^{\sigma_{n_m}}_{\sigma_{n_m+1}}\ J_{n_m} \\
\vdots\ \mathcal{C}_{n_{m+1}} & \vdots
\end{array}
$$

$$
\begin{array}{c}
\dfrac{\Phi_m, \neg A_m \qquad\qquad A_m, \Psi_m}{\Phi_m, \Psi_m}\ (\Sigma_{i_m})^{\sigma_{n_m+1}}\ K_m \\
\vdots\ \mathcal{C}_{n_{m+1}} \\
\dfrac{\Gamma_{n_{m+1}}}{\Gamma'_{n_{m+1}}}\ (c)^{\sigma_{n_{m+1}}}\ J_{n_{m+1}} \\
\vdots \\
\dfrac{\Gamma_{n-1}}{\Gamma'_{n-1}}\ (c)^{\sigma_{n-1}}_{\sigma}\ J_{n-1}
\end{array}
$$

$Fig.4$

where $\mathcal{R} = J_0, \ldots, J_{n-1}$ denotes a series of rules $(c)^{\sigma_p}_{\sigma_{p+1}}\, J_p$ with $\pi = \sigma_0, \sigma = \sigma_n$. $(\Pi_N\text{-rfl})\, H$ is resolved into a $(c)^\sigma_\rho\, J_n$ and a $(\Sigma_N)^\rho$ below J_{n-1}.

This series $\mathcal{R}$ is devided into intervals $\{\mathcal{R}_m = J_{n_{m-1}+1}, \ldots, J_{n_m} : m \leq l\}$ with an increasing sequence $n_{-1} + 1 = 0 \leq n_0 < n_1 < \cdots < n_l = n - 1\, (l \geq 0)$ of numbers so that

1. $\mathcal{R}_0 = J_0, \ldots, J_{n_0}$ is a chain $\mathcal{C}_{n_0}$ leading to J_{n_0}.

2. For $m < l$ $\mathcal{R}_{m+1} = J_{n_m+1}, \ldots, J_{n_{m+1}}$ is a tail of a chain $\mathcal{C}_{n_{m+1}} = J^{m+1}_0, \ldots, J^{m+1}_{n_m}, J_{n_m+1}, \ldots, J_{n_{m+1}}$ leading to $J_{n_{m+1}}$ such that the chain $\mathcal{C}_{n_{m+1}}$ passes through the left side of an inference rule $(\Sigma_{i_m})^{\sigma_{n_m+1}}\, K_m$ with $2 \leq i_m < N-1$, J_{n_m} is above the right uppersequent A_m, Ψ_m and $J^{m+1}_{n_m}$ is above the left uppersequent $\Phi_m, \neg A_m$ of K_m, resp. A_m is a Σ_{i_m} sentence. Each rule J^{m+1}_p for $p \leq n_m$ is again an inference rule $(c)^{\sigma_p}_{\sigma_{p+1}}$. K_m will be

a merging point of chains $\mathcal{C}_{n_{m+1}}$ and a new chain $\mathcal{C}_\rho = J_0, \ldots, J_{n-1}, J_n$ leading to $(c)_\rho\, J_n$.

3. There is no such a merging point below J_{n-1}, vz. there is no $(\Sigma_k)^\sigma$ with $1 < k < N-1$ such that J_{n-1} is in the right upper part of the inference rule and there exists a chain passing through its left side.

Set $N-1 \in In(\rho), rg_{N-1}(\rho) = \pi$ and $st_{N-1}(\rho)$ is the o.d. attached to the upper part of $(c)^\pi\, J_0$, where by the upper part we mean the part after resolving $(\Pi_N\text{-rfl})\, H$.

First consider the case $l = 0$, i.e., there is no merging point for the new chain $\mathcal{C}_\rho$ leading to the new J_n. Then set $In(\rho) = \{N-1\}$ and $pd_{N-1}(\rho) = \sigma$.

Suppose $l > 0$ in what follows. Then set $pd_{N-1}(\rho) = \sigma_{n_0+1}$, i.e., $pd_{N-1}(\rho)$ is the superscript of the first uppermost merging point $(\Sigma_{i_0})^{\sigma_{n_0+1}}\, K_0$.

In any cases we have $st_{N-1}(\rho) < st_{N-1}(pd_{N-1}(\rho))$, cf. (1). $st_i(\rho)$ always corresponds to the upper part of a $(c)^{rg_i(\rho)}$ in the chain $\mathcal{C}_\rho$ for $i \in In(\rho)$

2.2.1 The simplest case $N = 4$

Here suppose $N = 4$ and we determine the Q part of ρ. First set $2 \in In(\rho)$, vz. $In(\rho) = \{2, 3\}$ and $pd_2(\rho) = \sigma$. It remains to determine the o.d. $rg_2(\rho)$. In other words to specify a rule $(c)^{\sigma_q}\, J_q$ with $rg_i(\rho) = \sigma_q$.

Note that $i_m = 2$ for any m with $0 < m \leq l$ since $2 \leq i_m < N-1 = 3$ in this case. There are two cases to consider. First suppose there is a $p < n$ such that

1. $p > n_0$, i.e., $\sigma_{p+1} \prec_2 \sigma_{n_0+1} = pd_3(\rho)$ and
2. $2 \in In(\sigma_{p+1})$, i.e., there was a merging point of the chain leading to $(c)_{\sigma_{p+1}}\, J_p$.

Then pick the minimal q satisfying these two conditions, vz. the uppermost rule $(c)^{\sigma_q}_{\sigma_{q+1}}\, J_q$ below the first uppermost merging point $(\Sigma_{i_0})^{\sigma_{n_0+1}}\, K_0$ with $2 \in In(\sigma_{q+1})$. Then set

Case 1 $rg_2(\rho) = rg_2(\sigma_{q+1})$.

Otherwise set

Case 2 $rg_2(\rho) = \sigma = pd_2(\rho)$.

Consider the first case **Case 1** $rg_2(\rho) = rg_2(\sigma_{q+1}) \neq pd_2(\rho)$. From the definition we see $rg_2(\rho) = rg_2(\sigma_{q+1}) = pd_2(\sigma_{q+1}) = \sigma_q$. We have $\sigma_q = rg_2(\rho) \preceq_3 pd_3(\rho) = \sigma_{n_0+1}$. This follows from the minimality of q, i.e., $\forall t[n_0 < t < q \to 2 \notin In(\sigma_{t+1})]$ and hence $\forall t[n_0 < t < q \to \sigma_t = pd_2(\sigma_{t+1}) = pd_3(\sigma_{t+1})]$.

Furthermore q is minimal, i.e, σ_q is maximal in the following sense:

$$\begin{aligned}&\forall t[n_0 < t < n(\leftrightarrow pd_2(\rho) = \sigma \preceq_2 \sigma_{t+1} \prec_2 pd_3(\rho))\,\&\, rg_2(\sigma_{t+1})\downarrow \\ &\to rg_2(\sigma_{t+1}) \preceq_2 \sigma_q]\end{aligned} \tag{3}$$

In general we have the following fact.

Proposition 2.1 *Let $\mathcal{C} = J_0, \ldots, J_{n-1}$ be a chain leading to a $(c)^{\sigma_{n-1}}_{\sigma_n} J_{n-1}$. Each J_p is a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$ with $\sigma_0 = \pi$. Suppose that $2 \in In(\sigma_n)$ and the chain passes through the left side of a $(\Sigma_2)^{\sigma_p} K$ for a p with $0 < p < n$ so that J_{p-1} is in the left upper part of K and J_p is below K. Then $\sigma_q = rg_2(\sigma_n) \preceq_2 \sigma_p$, i.e., $q \geq p$.*

$$
\begin{array}{c}
\vdots\ \mathcal{C} \qquad\qquad\qquad\qquad\qquad\qquad \\
\dfrac{\Gamma_{p-1}}{\Gamma'_{p-1}}\ (c)^{\sigma_{p-1}}_{\sigma_p}\ J_{p-1} \qquad\qquad\qquad \\
\vdots \qquad\qquad\qquad\qquad \vdots \\
\dfrac{\Phi, \neg A^{\sigma_p} \qquad\qquad\qquad A^{\sigma_p}, \Psi}{\Phi, \Psi}\ (\Sigma_2)^{\sigma_p}\ K \\
\vdots\ \mathcal{C} \\
\dfrac{\Gamma_p}{\Gamma'_p}\ (c)^{\sigma_p}\ J_p \\
\vdots \\
\dfrac{\Gamma_{n-1}}{\Gamma'_{n-1}}\ (c)^{\sigma_{n-1}}_{\sigma}\ J_{n-1}
\end{array}
$$

This means that when, in $Fig.4$ a $(\Sigma_3)^{\sigma_t} K^3$ $(0 < t \leq n)$ in the new chain $\mathcal{C}_\rho = J_0, \ldots, J_{n-1}, J_n$ leading to $(c)^{\sigma}_{\rho} J_n$ is to be resolved into a $(\Sigma_2)^{\sigma_t} K^2$, then $t \leq q$, i.e., $rg_2(\rho) = \sigma_q \preceq_2 \sigma_t$. In other words any $(\Sigma_3)^{\sigma_t}$ with $q < t \leq n$, equivalently $(\Sigma_3)^{\sigma_t}$ which is below $(c)^{\sigma_q} J_q$ has to wait to be resolved, until the chain $\mathcal{C}_\rho$ will disapper by inversion.

For example consider, in $Fig.4$, an inference rule $(\Sigma_3)^{\sigma_t} K^3$ for $t = n_{m+1} + 1$. Its right cut formula is a $\Sigma_3^{\sigma_t}$ sentence C^{σ_t} and a descendent of a Σ_3 sentence C: a series of sentences from C to C^{σ_t} are in the chain $\mathcal{C}_{n_{m+1}} = J_0^{m+1}, \ldots, J_{n_m}^{m+1}, J_{n_m+1}, \ldots, J_{n_{m+1}}$ leading to $J_{n_{m+1}}$. Then the chain $\mathcal{C}_{n_{m+1}}$ passes through the left side of the inference rule $(\Sigma_{i_m})^{\sigma_{n_m+1}} K_m$ and hence K^3 will not be resolved until K_m will be resolved and its right upper part will disapper since we always perform rewritings of proof figure on the rightmosot branch. But then the chain $\mathcal{C}_\rho$ will disapper by inversion since it passes through the right side of K_m. In this way we see Proposition 2.1, cf. Lemma 5.7 in Section 5 for a full statemnt and a detailed proof.

(3) is seen from Proposition 2.1 and the minimality of q. Thus we have shown, cf. the conditions $(\mathcal{D}^Q.1)$ for $Od(\Pi_4)$ in [9] or Section 4,

$$rg_2(\sigma) = rg_2(pd_2(\rho)) \preceq_2 rg_2(\rho) \preceq_3 pd_3(\rho)$$

and

$$\forall t[rg_2(pd_2(\rho)) \preceq_2 \sigma_t \prec_2 \sigma_q \Rightarrow rg_2(\sigma_t) \preceq_2 \sigma_q].$$

Furthermore we have

$$st_2(\rho) < st_2(\sigma_{p+1}) < \sigma_q^+ \tag{4}$$

for the maximal p, vz. for the latest $(c)_{\sigma_{p+1}} J_p$ with $rg_2(\sigma_{p+1}) = \sigma_q \,\&\, 2 \in In(\sigma_{p+1})$.

Let $m < l$ denote the number such that $n_m < q \leq n_{m+1}$, i.e., J_q is a member of the tail $\mathcal{R}_{m+1} = J_{n_m+1}, \ldots, J_{n_{m+1}}$ of the chain $\mathcal{C}_{n_{m+1}}$. Then from Proposition 2.1 we see that J_p is also a member of $\mathcal{R}_{m+1}$ and further that J_q is a member of a chain $\mathcal{C}_p$ leading to J_p. Thus the upper part of $(c)^{\sigma_q} J_q$ corresponding to $st_2(\rho)$ is a result of perfoming several non-void rewritings to the upper part of a $(c)^{\sigma_q}$ which determined $st_2(\sigma_{p+1})$ when $(c)_{\sigma_{p+1}} J_p$ was introduced originally. This yields (4).

Thus we have established the conditions ($\mathcal{D}^Q$.1) in [9] or Section 4 for the newly introduced ρ.

Why we choose such a σ_q as $rg_2(\rho)$? First introducing $\sigma_q = rg_2(\rho)$ is meant to express the fact that σ_q is (iterated) Π_3-reflecting and it is responsible to $\Sigma_2^{\sigma_q}$ sentences occurring above a $(c)^{\sigma_q}$. Therefore even if there exists a σ_{p+1} above $pd_3(\rho)$, i.e., $p \leq n_0$ such that $2 \in In(\sigma_{p+1})$, we ignore these in determining $rg_2(\rho)$. Second in the **Case 1** the reason why we chose σ_q as the uppermost one is explained by Proposition 2.1: any $(\Sigma_3)^{\sigma_t}$ in the new chain $\mathcal{C}_\rho$ will not be resolved for $q < t \leq n$ until the chain $\mathcal{C}_\rho$ will disapper by inversion. Hence any σ_{q_1} with $rg_2(\sigma_{p_1+1}) = \sigma_{q_1} \prec_2 \sigma_q$ for some $p_1 \leq n$ will not be $rg_2(\kappa)$ for $\kappa \prec_2 \rho$ in the future. This means that a collapsing series $\{(c)_\kappa : rg_2(\kappa) = \sigma_{q_1}\}$ expressing the fact that σ_{q_1} is Π_3-reflecting is killed by introducing ρ such that $\rho \prec_2 \sigma_{q_1} \prec_2 \sigma_q = rg_2(\rho)$. Therefore once we introduce such a ρ, then we can ignore $rg_2(\sigma_{p_1+1}) = \sigma_{q_1}$ between $rg_2(\rho)$ and ρ.

2.2.2 The general case $N > 4$

Here suppose $N > 4$ and we determine the Q part of ρ, i.e., determine the set $In(\rho)$ and o.d.'s $pd_i(\rho), rg_i(\rho)$ for $i \in In(\rho)$ by referring $Fig.4$.

First set $i_0 \in In(\rho)$ where i_0 denotes the number such that the first merging point is a $(\Sigma_{i_0})^{\sigma_{n_0+1}} K_0$. Now let us assume inductively that for $k_0 \geq 0$ we have specified merging points $\{K_{m_k} : k \leq k_0\}$ so that $0 = m_0 < \cdots < m_{k_0}$, $N-1 > i_{m_0} > \cdots > i_{m_{k_0}} \geq 2$ and $\forall m \forall k < k_0 [m_k < m < m_{k+1} \to i_m \geq i_{m_k}]$, and have setted $\{i_{m_k} : k \leq k_0\} \subseteq In(\rho)$. Namely $K_{m_0}, \ldots, K_{m_{k_0}}$ is a series of merging points going downwards with decreasing indices i_{m_k} and K_{m_k} is the uppermost merging point with $i_{m_k} < i_{m_{k-1}}$ ($i_{m_{-1}} := N-1$).

If there exists an $m < l$ such that $m_{k_0} < m \,\&\, i_{m_{k_0}} > i_m \geq 2$, then let m denote the minimal one, vz. the uppermost merging point K_m below the latest one $K_{m_{k_0}}$ with $i_{m_{k_0}} > i_m$, and set $i_m \in In(\rho)$. Otherwise set

$$In(\rho) = \{i_{m_k} : k \leq k_0\} \cup \{N-1\}.$$

This completes a description of the set

$$\begin{aligned} In(\rho) &= \{N-1 = i_{m_{-1}}\} \cup \{i_{m_k} : 0 \leq k \leq k_1\} \\ &= \{N-1 = i_{m_{-1}} > i_{m_0} > \cdots > i_{m_{k_1}}\}. \end{aligned}$$

Observe that for $i < N-1$

$$i \in In(\rho) \Leftrightarrow \exists m < l [i_m = i \,\&\, \forall p < m (i_p \geq i)].$$

Now set $pd_{i_{m_k}}(\rho) = \sigma_{n_{m_{k+1}}+1}$ for $-1 \leq k \leq k_1$ with $m_{k_1+1} := l$, vz. the merging point K_{m_k} chosen for $i_{m_k} \in In(\rho)$ is a $(\Sigma_{i_{m_k}})^{pd_{i_{m_{k-1}}}(\rho)}$ for $0 \leq k \leq k_1$ and $pd_2(\rho) = pd_{i_{m_{k_1}}}(\rho) = \sigma_{n_l+1} = \sigma_n = \sigma$. Observe that for any i with $2 \leq i \leq N-1$ there exists an $m(i) \leq l$ such that $pd_i(\rho) = \sigma_{n_{m(i)}+1}$ and this $m(i)$ is the minimal m for which $i_m < i$.

It remains to determine the o.d.'s $rg_i(\rho)$ for $N-1 \neq i = i_{m_k} \in In(\rho)$. As in the case $N=4$ there are two cases to consider. First suppose there is a $p < n$ such that

1. $\rho \prec_i \sigma_{p+1} \prec_i \sigma_{n_{m_k}+1} = pd_{i_{m_{k-1}}}(\rho) = pd_{i+1}(\rho)$ and
2. $i \in In(\sigma_{p+1})$.

Then pick the minimal p satisfying these two conditions, vz. the uppermost rule $(c)_{\sigma_{p+1}} J_p$ below the merging point $(\Sigma_i)^{pd_{i+1}(\rho)} K_{m_k}$ with $\sigma_{p+1} \prec_i pd_{i+1}(\rho) \,\&\, i \in In(\sigma_{p+1})$. Then set

Case 1 $rg_i(\rho) := \sigma_q := rg_i(\sigma_{p+1})$.

Otherwise set

Case 2 $rg_i(\rho) = pd_i(\rho)$.

In general we have the following fact.

Proposition 2.2 *Let $\mathcal{C} = J_0, \ldots, J_{n-1}$ be a chain leading to a $(c)^{\sigma_{n-1}}_{\sigma_n} J_{n-1}$. Each J_p is a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$ with $\sigma_0 = \pi$. Suppose that the chain passes through the left side of a $(\Sigma_j)^{\sigma_p} K$ for a p with $0 < p < n$ and a $j \geq i$ so that J_{p-1} is in the left upper part of K and J_p is below K. Then $\sigma_n \prec_i \sigma_p$ and if further $N-1 \neq i \in In(\sigma_n)$, then $\sigma_q = rg_i(\sigma_n) \preceq_i \sigma_p$, cf. the figure in Proposition 2.1.*

Let us expalin this Proposition 2.2 using the new chain $\mathcal{C}_\rho = J_0, \ldots, J_{n-1}, J_n$ leading to $(c)^\sigma_\rho J_n$, cf. $Fig.4$. When a $(\Sigma_{j+1})^{\sigma_t} K^{j+1}\,(0 < t \leq n)$ in the new chain $\mathcal{C}_\rho$ is to be resolved, a $(\Sigma_j)^{\sigma_s} K^j$ is introduced at a point below K^{j+1}. The point and $s \geq t$ is determined as the lowest position as far as we can lower a rule $(\Sigma_j)^{\sigma_t}$, cf. Definition 5.5 in Section 5. For example when K^{j+1} is the rule $(\Sigma_{i_m})^{\sigma_{n_m+1}} K_m$ in $Fig.4$, let m_1 denote the minimal m_1 such that $i_{m_1} < i_m$ and we introduce a new $(\Sigma_{i_m-1})^{\sigma_{n_{m_1+1}}}$ ($s = n_{m_1+1}$) between the rules $(c)_{\sigma_{n_{m_1+1}}} J_{n_{m_1}}$ and $(\Sigma_{i_{m_1}})^{\sigma_{n_{m_1}+1}} K_{m_1}$. Observe that the new (Σ_{i_m-1}) together with $(\Sigma_{i_{m_2}}) K_{m_2}\,(m < m_2 < m_1)$ by inversion will be merging points for the next chain leading to a $(c)^\rho$.

Let us consider the case when the $(\Sigma_{i_m-1})^{\sigma_{n_{m_1+1}}}$ is the rule $(\Sigma_j)^{\sigma_p} K$ in the Proposition 2.2: $j = i_m - 1 \,\&\, p = n_{m_1+1}$. Also put $pd_i(\rho) = \sigma_{n_{m(i)}+1}$, where $m(i)$ denotes the minimal $m(i)$ such that $i_{m(i)} < i$. Then $i \leq j = i_m - 1$. By Proposition 2.3 below we see that $i_m < i_{m_3}$ for any $m_3 < m$, i.e., any merging point $(\Sigma_{i_{m_3}}) K_{m_3}$ above $(\Sigma_{i_m}) K^{j+1} = K_m$ has larger index since we are assuming that K_m is to be resolved. Therefore $m(i) \geq m_1$, i.e., the merging point $(\Sigma_{i_{m(i)}})^{\sigma_{n_{m(i)+1}}} K_{m(i)}$ determining $pd_i(\rho)$ is equal to or below the merging

point $(\Sigma_{i_{m_1}})^{\sigma_{n_{m_1}+1}} K_{m_1}$. In the former case we have $pd_i(\rho) = \sigma_{n_{m(i)}+1} = \sigma_{n_{m_1}+1} = \sigma_p$ and hence $\rho \prec_i \sigma_p$. In the latter case we have $i_{m_3} \geq i$ for $m_1 \leq m_3 < m(i)$. Thus we see $\rho \prec_i \sigma_p$ inductively. This shows the first half of Proposition 2.2.

Now assume $N-1 \neq i \in In(\sigma_n)$ and show $rg_i(\rho) \preceq_i \sigma_p$. Consider the **Case 1**, vz. $\sigma_q = rg_i(\rho) \neq pd_i(\rho)$. Let p_0 denote the minimal p_0 such that $\rho \prec_i \sigma_{p_0+1} \prec_i pd_{i+1}(\rho)$ and $i \in In(\sigma_{p_0+1})$. By the definition we have $\sigma_q = rg_i(\rho) = rg_i(\sigma_{p_0+1})$.

Let $m(i+1) < m(i)$ denote the number such that $pd_{i+1}(\rho) = \sigma_{n_{m(i+1)}+1}$. Then by $i \in In(\rho)$ we have $i_{m(i+1)} = i \leq i_{m_1}$, i.e., $m_1 \leq m(i+1) < m(i)$ and hence $pd_{i+1}(\rho) \leq \sigma_{n_{m_1}+1} = \sigma_p$. On the other we have $\rho \prec_i \sigma_q = rg_i(\rho)$ by the definition and $\rho \prec_i \sigma_p$ by the first half of the Proposition 2.2. Hence it suffices to show $\sigma_q \leq \sigma_p$ since the set $\{\tau : \rho \prec_i \tau\}$ is linearly ordered by $\prec_i$. Now we see $\sigma_q = rg_i(\rho) = rg_i(\sigma_{p_0+1}) \preceq_i pd_{i+1}(\rho)$ inductively, i.e., by using Proposition 2.2 for smaller parts. Thus we get $\sigma_q \preceq_i pd_{i+1}(\rho) \leq \sigma_p$. This shows the second half of Proposition 2.2.

Further we have the following fact.

Proposition 2.3 *Let $\mathcal{C} = J_0, \ldots, J_{n-1}$ be a chain leading to a $(c)^{\sigma_{n-1}}_{\sigma_n} J_{n-1}$. Each J_k is a rule $(c)^{\sigma_k}_{\sigma_{k+1}}$ with $\sigma_0 = \pi$. Suppose that the chain $\mathcal{C}$ passes through the left side of a $(\Sigma_j)^{\sigma_p} K^{lw}$ for a p with $0 < p < n$ so that J_{p-1} is in the left upper part of K^{lw} and J_p is below K^{lw}. Let $\mathcal{D} = I_0, \ldots, I_{m-1}\,(m \geq p)$ be a chain leading to a $(c)^{\sigma_{m-1}}_{\sigma_m} I_{m-1}$. Each I_k is a rule $(c)^{\tau_k}_{\tau_{k+1}}$ such that $\tau_k = \sigma_k$ for $0 \leq k < \min\{n, m\}$. Suppose that the chain $\mathcal{D}$ passes through the left side of a $(\Sigma_i)^{\sigma_k} K^{up}$ for a k with $0 < k < p$ so that I_{k-1} is in the left upper part of K^{up} and I_k is below K^{up}. Further assume the rule $(c)_{\sigma_p} I_{p-1}$ is in the right upper part of $(\Sigma_j)^{\sigma_p} K^{lw}$ and $i \leq j$.*

Then the upper K^{up} foreruns the lower K^{lw}, i.e., analyses of K^{up} have to precede ones of K^{lw}.

Let us explain Proposition 2.3 by referring $Fig.4$: $\mathcal{C}$ is the new chain $\mathcal{C}_\rho$, K^{lw} is the new $(\Sigma_{i_m-1})^{\sigma_{n_{m_1}+1}}$ which is resulted from $(\Sigma_{i_m})^{\sigma_{n_m+1}} K_m$ with $m = l-1$, i.e., the resolved rule K_{l-1} is the lowest merging point. Then K^{lw} is a $(\Sigma_{i_m-1})^{\sigma}$ with $m_1 = l$. Further $\mathcal{D}$ is the chain $\mathcal{C}_{n_{m+1}} = J^{m+1}_0, \ldots, J^{m+1}_{n_m}, J_{n_m+1}, \ldots, J_{n_{m+1}}$ leading to the last member $(c)_\sigma J_{n-1}\,(n-1 = n_{m+1} = n_l)$ of the series $\mathcal{R}$. Then the last member $(c)_\sigma J_{n-1}$ is in the right upper part of $(\Sigma_{i_m-1})^\sigma K^{lw}$. Let I be a rule $(\Sigma_{i+1})^\tau$ such that the chain $\mathcal{D}$ passes through its right side. Suppose the rule I in the chain $\mathcal{D}$ is resolved and produces a $(\Sigma_i)^{\sigma_k} K^{up}$ for a k with $0 < k < n$ so that the chain $\mathcal{D}$ passes through the left side of K^{up}.

$$
\dfrac{
\begin{array}{c}
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{cc}\vdots & \vdots\ \mathcal{D} \\ \Pi, \neg B & B, \Lambda, C_m^{\tau}\end{array}}{\Pi, \Lambda, C_m^{\tau}}\ (\Sigma_{i+1})^{\tau} \\
\vdots\ \mathcal{D} \\
\Phi_m, C_m^{\sigma_{n_m+1}}
\end{array}
\qquad
\begin{array}{c}
\vdots\ \mathcal{C} \\
\neg C_m^{\sigma_{n_m+1}}, \Psi_m, \neg C^{\sigma_{n_m+1}}
\end{array}
}{\Phi_m, \Psi_m, \neg C^{\sigma_{n_m+1}}}\ (\Sigma_{i_m})^{\sigma_{n_m+1}}\ K_m \\
\vdots\ \mathcal{D}, \mathcal{C} \\
\dfrac{\Gamma_{n-1}, \neg C^{\sigma_{n-1}}}{\Gamma'_{n-1}, \neg C^{\sigma}}\ (c)_{\sigma}^{\sigma_{n-1}}\ J_{n-1} \\
\vdots \\
\Phi, \neg C^{\sigma}
\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{cc}\vdots & \vdots\ \mathcal{D} \\ \Pi, \neg B & B, C^{\tau}, \Lambda\end{array}}{C^{\tau}, \Pi, \Lambda}\ (\Sigma_{i+1})^{\tau}\ I \\
\vdots \\
\dfrac{C^{\sigma_{n_m+1}}, \Phi_m}{C^{\sigma_{n_m+1}}, \Phi_m, \Psi_m} \\
\vdots\ \mathcal{D} \\
\dfrac{C^{\sigma_{n-1}}, \Gamma_{n-1}}{C^{\sigma}, \Gamma'_{n-1}}\ (c)_{\sigma}^{\sigma_{n-1}}\ J_{n-1} \\
\vdots \\
C^{\sigma}, \Psi
\end{array}
}{
\begin{array}{c}
\Phi, \Psi \\
\vdots\ \mathcal{C} \\
\dfrac{\Gamma_n}{\Gamma'_n}\ (c)_{\rho}^{\sigma}\ J_n
\end{array}
}\ (\Sigma_{i_m-1})^{\sigma}\ K^{lw}
$$

Fig.5

where $\neg A_m \equiv C_m^{\sigma_{n_m+1}} \equiv \forall x < \sigma_{n_m+1} C_0(x)$ and $C^{\sigma_{n_m+1}} \equiv C_0(\alpha)$ for an $\alpha < \sigma_{n_m+1}$.

$$
\dfrac{
\begin{array}{c}
\vdots\ \mathcal{D}, \mathcal{C} \\
\dfrac{\Gamma_{n-1}, \neg C^{\sigma_{n-1}}}{\Gamma'_{n-1}, \neg C^{\sigma}}\ (c)_{\sigma}^{\sigma_{n-1}}\ J_{n-1} \\
\vdots \\
\Phi, \neg C^{\sigma}
\end{array}
\qquad
\begin{array}{c}
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{cc}\vdots & \vdots\ \mathcal{D} \\ \Pi, \neg B & B, C^{\tau}, \Lambda, B_1^{\tau}\end{array}}{C^{\tau}, \Pi, \Lambda, B_1^{\tau}}\ (\Sigma_{i+1})^{\tau}\ I \\
\vdots\ \mathcal{D} \\
C^{\sigma_k}, \Lambda_1, B_1^{\sigma_k}
\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots \\ \Pi, \neg B_1^{\tau}\end{array}}{\neg B_1^{\tau}, \Pi, \Lambda} \\
\vdots \\
\neg B_1^{\sigma_k}, \Pi_1
\end{array}
}{C^{\sigma_k}, \Lambda_1, \Pi_1}\ (\Sigma_i)^{\sigma_k}\ K^{up} \\
\vdots\ \mathcal{D} \\
\dfrac{C^{\sigma_{n-1}}, \Gamma_{n-1}}{C^{\sigma}, \Gamma'_{n-1}}\ (c)_{\sigma}^{\sigma_{n-1}}\ J_{n-1} \\
\vdots \\
C^{\sigma}, \Psi
\end{array}
}{
\begin{array}{c}
\Phi, \Psi \\
\vdots\ \mathcal{C} \\
\dfrac{\Gamma_n}{\Gamma'_n}\ (c)_{\rho}^{\sigma}\ J_n
\end{array}
}\ (\Sigma_{i_m-1})^{\sigma}\ K^{lw}
$$

Fig.6

We show, in *Fig*.6, no ancestor of the right cut formula C^{σ} of K^{lw} is in the right upper part of K^{up} in order to see that K^{up} foreruns K^{lw}. It sufices to see that, in *Fig*.5, no ancestor of the right cut formula C^{σ} of K^{lw} is in the left upper part of the resolved rule $(\Sigma_{i+1})^{\tau}\ I$. Any ancestor of the right cut formula C^{σ} of K^{lw} comes from the left cut formula $\neg A_m \equiv C_m^{\sigma_{n_m+1}}$ of $(\Sigma_{i_m})^{\sigma_{n_m+1}}\ K_m$ and any ancestor of the latter is in the chain $\mathcal{D}$, which in turn passes through

the right side of $(\Sigma_{i+1})^{\tau}\, I$. Thus any ancestor of the right cut formula C^{σ} of K^{lw} is in the right upper part of I in $Fig.5$, a fortiori, in the left upper part of K^{up} in $Fig.6$. This shows Proposition 2.3.

For full statements and proofs of Propositions 2.2, 2.3, see Lemmata 5.7, the proviso **(uplw)** in Definition 5.8 in Section 5 and the case **M7.2** in Section 6.

From Propositions 2.2, 2.3 we see that the conditions $(\mathcal{D}^Q.1)$ for $Od(\Pi_N)$ in [9] or Section 4 are enjoyed with respect to the Q part of ρ as for the case $N = 4$. A set-theoretic meaning and a wellfoundedness proof of $Od(\Pi_N)$ are derived from these conditions on o.d.'s as we saw in [8] and [9].

Consider a rule (Σ_j) in the chain $\mathcal{C}_\rho$ for $j \geq i \in In(\rho)$ which is below $(\Sigma_{i_{m(i)}})^{pd_i(\rho)}\, K_{m(i)}\, (i_{m(i)} < i)$. Then from Proposition 2.3 we see that analyses of such a (Σ_j) have to follow ones of the rule $(\Sigma_{i_{m(i)}})^{pd_i(\rho)}\, K_{m(i)}$. Thus when such a reversal happens, the lower rule with greater indices $(j > i_{m(i)})$ is dead and we can ignore it. The o.d. $pd_i(\rho)$ and the rule $(c)_{pd_i(\rho)}\, J_{n_{m(i)}}$ is the predecessor of the o.d. ρ and the rule $(c)_\rho$ with respect to i: any member $(c)_\kappa$ of the chain $\mathcal{C}_\rho$ with $\rho < \kappa < pd_i(\rho)$ is irrelevant to the fact that $pd_i(\rho)$ and $rg_i(\rho)$ are iterated Π_i-reflecting. But the member may be relevant to Π_j-reflection for $j < i$. This motivates the definitions of $In(\rho)$ and $pd_i(\rho)$. A series $\kappa_n \prec_i \kappa_{n-1} \prec_i \cdots \prec_i \kappa_0$ expesses a possible stepping down for the fact that κ_0 is an iterated Π_i-reflecting ordinal. Degrees of iterations are measured by an ordinal $\nu < \kappa^+$ with $\kappa = rg_i(\kappa_0), \nu = st_i(\kappa_0)$ (and by predecessors of $rg_i(\kappa_0)$) as we saw in [8] and [9]. Therefore we seach only for o.d.'s σ_{p+1} with $\rho \prec_i \sigma_{p+1}$ in determing the o.d. $rg_i(\rho) = rg_i(\sigma_{p+1})$.

In the **Case 1** the reason why we chose σ_q as the uppermost one is explained by Propositions 2.2, 2.3 as in the case $N = 4$.

Now details follow.

3 The theory T_N for Π_N-reflecting ordinals

In this section a theory T_N of Π_N-reflecting ordinals is defined.

Let T_0 denote the base theory defined in [5]. $\mathcal{L}_1$ denotes the language of T_0. Recall that $\mathcal{L}_1 = \mathcal{L}_0 \cup \{R^{\mathcal{A}}, R^{\mathcal{A}}_{<} : \mathcal{A} \text{ is a } \Delta_0 \text{ formula in } \mathcal{L}_0 \cup \{X\}\}$ with $\mathcal{L}_0 = \{0, 1, +, -, \cdot, q, r, \max, j, ()_0, ()_1, =, <\}$. $R^{\mathcal{A}}, R^{\mathcal{A}}_{<}$ are predicate constants for inductively defined predicates. The axioms and inference rules in T_0 are designed for this language $\mathcal{L}_1$.

The *language* $\mathcal{L}(\mathrm{T}_N)$ of the theory T_N is defined to be $\mathcal{L}_1 \cup \{\Omega\}$ with an individual constant Ω.

The *axioms* of T_N are the same as for the theory T_3 in [6], i.e., are obtained from those of T_{22} in [5] by deleting the axiom $\Gamma, Ad(\Omega)$. Thus the axioms Γ, Λ_f for the closure of Ω under the function f in $\mathcal{L}_0$ are included as mathematical axiom in T_N.

The *inference rules* in T_N are obtained from T_0 by adding the following rules

(Π_N-rfl) and (Π_2^Ω-rfl).

$$\frac{\Gamma, A \quad \neg\exists z(t_0 < z \land A^z), \Gamma}{\Gamma} \ (\Pi_N\text{-rfl})$$

where $A \equiv \forall x_N \exists x_{N-1} \cdots Q x_1 B(x_N, x_{N-1}, \ldots, x_1, t_0)$ is a Π_N formula.

$$\frac{\Gamma, A^\Omega \quad \neg\exists z(t < z < \Omega \land A^z), \Gamma \quad \Gamma, t < \Omega}{\Gamma} \ (\Pi_2^\Omega\text{-rfl})$$

where $A \equiv \forall x \exists y B(x, y, t)$ is a Π_2 formula.

Concepts related to proof figures, principal or auxiliary formulae, pure variable condition, branch, etc. are defined exactly as in Section 2 of [5].

4 The system $Od(\Pi_N)$ of ordinal diagrams

In this section first let us recall briefly the system $Od(\Pi_N)$ of ordinal diagrams (abbreviated by o.d.'s) in [9].

Let $0, \varphi, \Omega, +, \pi$ and d be distinct symbols. Each o.d. in $Od(\Pi_N)$ is a finite sequence of these symbols. φ is the Veblen function. Ω denotes the first recursively regular ordinal ω_1^{CK} and π the first Π_N-reflecting ordinal.

The set $Od(\Pi_N)$ is classified into subsets R, SC, P according to the intended meanings of o.d.'s. P denotes the set of additive principal numbers, SC the set of strongly critical numbers and R the set of recursively regular ordinals (less than or equal to π). If $\pi > \sigma \in R$, then σ^+ denotes the next recursively regular diagram to σ.

Recall that $K\alpha$ denotes the finite set of o.d.'s defined as follows.

1. $K0 = \emptyset$.
2. $K(\alpha_1 + \cdots + \alpha_n) = \bigcup\{K\alpha_i : 1 \le i \le n\}$
3. $K\varphi\alpha\beta = K\alpha \cup K\beta$
4. $K\alpha = \{\alpha\}$ otherwise, i.e., $\alpha \in SC$.

Definition 4.1 *1.* $\mathcal{D}_\sigma(\alpha) \subseteq \mathcal{D}_\sigma$.

   (a) $\mathcal{D}_\sigma(\alpha) = \emptyset$ if $\alpha \in \{0, \Omega, \pi\}$.

   (b) $\mathcal{D}_\sigma(\alpha) = \mathcal{D}_\sigma(K\alpha)$ if $\alpha \notin SC$.

   (c) If $\alpha \in \mathcal{D}_\tau$,

$$\mathcal{D}_\sigma(\alpha) = \begin{cases} \mathcal{D}_\sigma(\{\tau\} \cup c(\alpha)) & \text{if } \tau > \sigma \\ \{\alpha\} \cup \mathcal{D}_\sigma(c(\alpha)) & \text{if } \tau = \sigma \\ \mathcal{D}_\sigma(\tau) & \text{if } \tau < \sigma \end{cases}$$

2. $\mathcal{B}_\sigma(\alpha) = \max\{b(\beta) : \beta \in \mathcal{D}_\sigma(\alpha)\}$.

3. $\mathcal{B}_{>\sigma}(\alpha) = \max\{\mathcal{B}_\tau(\alpha) : \tau > \sigma\}$.

For an o.d. α set

$$\alpha^+ = \min\{\sigma \in R \cup \{\infty\} : \alpha < \sigma\}.$$

For $\sigma \in R$, $\mathcal{D}_\sigma \subseteq SC$ denotes the set of o.d.'s of the form $\rho = d_\sigma^q \alpha$ with a (possibly empty) list q, where the following condition has to be met:

$$\mathcal{B}_{>\sigma}(\{\sigma, \alpha\} \cup q) < \alpha \tag{5}$$

α is the *body* of $d_\sigma^q \alpha$.

If q is not empty, then $d_\sigma^q \alpha \in \mathcal{D}^Q$ by definition. Its Q *part* $Q(d_\sigma^q \alpha) = q = \overline{\nu\kappa\tau j}$ denotes a sequence of quadruples $\nu_m \kappa_m \tau_m j_m$ of length $l+1$ $(0 \leq l)$ such that

1. $2 \leq j_0 < j_1 < \cdots < j_l = N-1$,
2. $\kappa_l = \pi, \kappa_m \in R \mid \pi \, (m < l) \, \& \, \sigma \preceq \kappa_m \, (m \leq l)$,
3. $\nu_l \in Od(\Pi_N)$,
$$\sigma = \pi \Rightarrow \nu_l \leq \alpha \tag{6}$$
and
$$m < l \Rightarrow \nu_m < \kappa_m^+ \tag{7}$$
4. $\tau_0 = \sigma, \tau_m \in \{\pi\} \cup \mathcal{D}^Q, \sigma \preceq \tau_m \, (m \leq l)$ and
$$\tau_l = \pi \Rightarrow \sigma = \pi \tag{8}$$

From $q = Q(\rho)$ define

1. $in_j(\rho) = st_j(\rho) rg_j(\rho)$ (a pair) and $pd_j(\rho)$: Given j with $2 \leq j < N$, put $m = \min\{m \leq l : j \leq j_m\}$.
2. $pd_j(\rho) = \tau_m$.
3. $\exists m \leq l(j = j_m)$: Then $st_j(\rho) = \nu_m$, $rg_j(\rho) = \kappa_m$.
4. Otherwise: $in_j(\rho) = in_j(pd_j(\rho)) = in_j(\tau_m)$. If $in_j(\tau_m) = \emptyset$, then set $st_j(\rho) \uparrow, rg_j(\rho) \uparrow$.
5. $In(\rho) = \{j_m : m \leq l\}$.

Observe that

$$\pi < \beta \in q = Q(\rho) \Rightarrow \beta = \nu_l = st_{N-1}(\rho) \tag{9}$$

The relation $\alpha \prec_i \beta$ is the transitive closure of the relation $pd_i(\alpha) = \beta$.

In [9] we impose several conditions on a diagram of the form $\rho = d_\sigma^q \alpha$ to be in $Od(\Pi_N)$. For $\alpha \in Od(\Pi_N), q \subseteq Od(\Pi_N) \, \& \, \sigma \in R \setminus \{\Omega\}$, $\rho = d_\sigma^q \alpha \in Od(\Pi_N)$ if the following conditions are fulfilled besides (5);

($\mathcal{D}^Q$.1) Assume $i \in In(\rho)$. Put $\kappa = rg_i(\rho)$. Then

($\mathcal{D}^Q$.11) $in_i(rg_i(\rho)) = in_i(pd_{i+1}(\rho))$, $rg_i(\rho) \preceq_i pd_{i+1}(\rho)$ and $pd_i(\rho) \neq pd_{i+1}(\rho)$ if $i < N-1$.
Also $pd_i(\rho) \preceq_i rg_i(\rho)$ for any i.

($\mathcal{D}^Q$.12) One of the following holds:

($\mathcal{D}^Q$.12.1) $rg_i(\rho) = pd_i(\rho) \,\&\, \mathcal{B}_{>\kappa}(st_i(\rho)) < b(\alpha_1)$ with $\rho \preceq \alpha_1 \in \mathcal{D}_\kappa$.

($\mathcal{D}^Q$.12.2) $rg_i(\rho) = rg_i(pd_i(\rho)) \,\&\, st_i(\rho) < st_i(pd_i(\rho))$.

($\mathcal{D}^Q$.12.3) $rg_i(pd_i(\rho)) \prec_i \kappa \,\&$
$\forall \tau (rg_i(pd_i(\rho)) \preceq_i \tau \prec_i \kappa \to rg_i(\tau) \preceq_i \kappa) \,\&\, st_i(\rho) < st_i(\sigma_1)$ with

$$\sigma_1 = \min\{\sigma_1 : rg_i(\sigma_1) = \kappa \,\&\, pd_i(\rho) \prec_i \sigma_1 \prec_i \kappa\}$$

and such a σ_1 exists.

($\mathcal{D}^Q$.2)

$$\forall \kappa \leq rg_i(\rho)(K_\kappa st_i(\rho) < \rho) \tag{10}$$

for $i \in In(\rho)$.

We set $Q(d_\sigma \alpha) = \emptyset$, i.e., $d_\sigma^\emptyset \alpha = d_\sigma \alpha$.

The order relation $\alpha < \beta$ on $\mathcal{D}_\sigma$ is defined through finite sets $K_\tau \alpha$ for $\tau \in R, \alpha \in Od(\Pi_N)$, and the latter is defined through the relation $\alpha \prec \beta$, which is the transitive closure of the relation $\alpha \in \mathcal{D}_\beta$. Thus $\alpha \prec_2 \beta \Leftrightarrow \alpha \prec \beta$.

For $\rho = d_\tau^q \alpha$ $c(\rho) = \{\tau, \alpha\} \cup q$ and

$$K_\sigma \rho = \begin{cases} K_\sigma(\{\tau\} \cup c(\rho)) = K_\sigma\{\tau, \alpha\} \cup q, & \sigma < \tau \\ K_\sigma \tau, & \tau < \sigma \,\&\, \tau \not\preceq \sigma \end{cases}$$

The following Proposition 4.1 is shown in [9].

Proposition 4.1 *1. The finite set $\{\tau : \sigma \prec_i \tau\}$ is linearly ordered by $\prec_i$.*
In the following assume $\kappa = rg_i(\rho) \downarrow$.

2. $\rho \prec_i rg_i(\rho)$.

3. $\rho \prec_i \sigma \prec_i \tau \,\&\, in_i(\rho) = in_i(\tau) \Rightarrow in_i(\rho) = in_i(\sigma)$.

4. $\rho \prec_i \tau \prec_i rg_i(\rho) \Rightarrow rg_i(\tau) \preceq_i rg_i(\rho)$.

Definition 4.2 For o.d.'s α, σ with $\sigma \in R$,

$$\mathcal{K}_\sigma(\alpha) := \max K_\sigma \alpha.$$

The following lemmata are seen as in [5].

Lemma 4.1 *Suppose $\mathcal{B}_{>\kappa}(\alpha_i) < \alpha_i$ for $i = 0, 1$, and $\alpha_0 < \alpha_1$. Then*

$$\tau > \kappa \Rightarrow d_\tau \alpha_i \in Od(\Pi_N) \,\&\, d_\tau \alpha_0 < d_\tau \alpha_1.$$

Lemma 4.2 *For $\alpha, \beta, \sigma \in Od(\Pi_N)$ with $\sigma \in R|\pi$ assume $\forall \tau < \pi[\mathcal{B}_\tau(\beta) \leq \mathcal{B}_\tau(\alpha)]$, and put $\gamma = \max\{\mathcal{B}_\pi(\beta), \mathcal{B}_{>\sigma}(\{\sigma, \alpha\})\} + \omega^\beta$. Then $\mathcal{B}_{>\sigma}(\{\sigma, \gamma, \gamma + \mathcal{K}_\sigma(\alpha)\}) < \gamma$, and hence (5) is fulfilled for $d_\sigma \gamma, d_\sigma(\gamma + \mathcal{K}_\sigma(\alpha)) \in Od(\Pi_N)$.*

5 The system T_{Nc}

In this section we extend T_N to a formal system T_{Nc}. The *universe* $\pi(\mathrm{T}_N)$ of the theory T_N is defined to be the o.d. $\pi \in Od(\Pi_N)$. The language is expanded so that individual constants c_α for o.d.'s $\alpha \in Od(\Pi_N) \mid \pi$ are included. Inference rules $(c)^\sigma$ are added. To each proof P in T_{Nc} an o.d. $o(P) \in Od(\Pi_N) \mid \Omega$ is attached. *Chains* are defined to be a consecutive sequence of rules (c). *Proofs* in T_{Nc} defined in Definition 5.8 are proof figures enjoying some provisos and obtained from given proofs in T_N by operating rewriting steps. Some lemmata for proofs are established. These are needed to verify that rewrited proof figures enjoy these provisos.

The *language* $\mathcal{L}_{Nc}$ of T_{Nc} is obtained from the language $\mathcal{L}(\mathrm{T}_N)$ by adding individual constants c_α for each o.d. $\alpha \in Od(\Pi_N)$ such that $1 < \alpha < \pi \,\&\, \alpha \neq \Omega$. We identify the constant c_α with the o.d. α.

In what follows $A, B, \ldots$ denote formulae in $\mathcal{L}_{Nc}$ and $\Gamma, \Delta, \ldots$ sequents in $\mathcal{L}_{Nc}$.

The *axioms* of T_{Nc} are obtained from those of T_N as in [5].

Complexity measures $\deg(A), \mathrm{rk}(A)$ of formulae A are defined as in [5] by replacing the universe $\pi(\mathrm{T}_{22}) = \mu$ by $\pi(\mathrm{T}_N) = \pi$.

Also the sets $\Delta_0^\sigma, \Sigma_i^\sigma$ of formulae are defined as in [5]. Recall that for a bounded formula A and a multiplicative principal number $\alpha \leq \pi$, we have $A \in \Delta^\alpha \Leftrightarrow \deg(A) < \alpha$.

Definition 5.1

$$\deg_N(A) := \begin{cases} \deg(A) + N - 1 & \text{if } A \text{ is a bounded formula} \\ \deg(A) & \text{otherwise} \end{cases}$$

Note that

$$\deg_N(A) \notin \{\alpha + i : i < N - 1, \alpha < \pi \text{ is a limit o.d.}\}$$

The *inference rules* of T_{Nc} are obtained from those of T_N by adding the following rules $(h)^\alpha\,(\alpha \in \{\alpha : \pi \leq \alpha < \pi + \omega\} \cup \{0, \Omega\})$, $(c\Pi_2)^\Omega_{\alpha_1}$, $(c\Sigma_1)^\Omega_{\alpha_1}$, $(c\Pi_N)^\sigma_\tau$, $(c\Sigma_{N-1})^\sigma_\tau$ for each $\sigma \in R \subseteq Od(\Pi_N) \,\&\, \sigma \neq \Omega$ and $(\Sigma_i)^\sigma$ for each $\sigma \in R \subseteq Od(\Pi_N) \,\&\, \sigma \notin \{\Omega, \pi\}$ and $i = 1, 2, \ldots, N$. The rule $(h)^\alpha$, $(c\Pi_2)^\Omega_{\alpha_1}$ and $(c\Sigma_1)^\Omega_{\alpha_1}$ are the same as in [6]. We write (w) for $(h)^0$.

1.

$$\frac{\Gamma, A^\sigma}{\Gamma, A^\tau}\ (c\Pi_N)^\sigma_\tau$$

where

(a) $A \equiv \forall x_N \exists x_{N-1} \cdots Q x_1 B$ is a Π_N-sentence with a Δ^τ-matrix B,

(b) $\tau \in \mathcal{D}_\sigma$ with the *body* $\alpha = b(\tau)$ of the rule and

(c) the formula A^τ in the lowersequent is the *principal formula* of the rule and the formula A^σ in the uppersequent is the *auxiliary formula* of the rule, resp. Each formula in Γ is a *side formula* of the rule.

2.
$$\frac{\Gamma, \Lambda^\sigma}{\Gamma, \Lambda^\tau}\ (c\Sigma_{N-1})^\sigma_\tau$$

where

(a) Λ is a nonempty set of unbounded Π_N-sentences with Δ^τ-matrices.

(b) $\tau \in \mathcal{D}_\sigma$ with the *body* $\alpha = b(\tau)$ of the rule and

(c) each formula in Γ is a *side formula* of the rule.

3.
$$\frac{\Gamma, \neg A^\sigma \quad A^\sigma, \Lambda}{\Gamma, \Lambda}\ (\Sigma_i)^\sigma$$

where $1 \leq i \leq N$ and A^σ is a genuine Σ_i^σ-sentence, i.e., $A^\sigma \in \Sigma_i^\sigma$ and $A^\sigma \not\in \Pi^\sigma_{i-1} \cup \Sigma^\sigma_{i-1}$.

A^σ $[\neg A^\sigma]$ is said to be the *right* [*left*] *cut formula* of the rule $(\Sigma_i)^\sigma$, resp.

The rules $(c\Pi_2)^\Omega$ and $(c\Pi_N)$ are *basic rules* but not the rules $(h)^\alpha$, $(c\Sigma_1)^\Omega$, $(c\Sigma_{N-1})^\sigma$ and $(\Sigma_i)^\sigma$.

A *preproof* in T_{Nc} is a proof in T_{Nc} in the sense of [5], i.e., a proof tree built from axioms and inference rules in T_{Nc}. The underlying tree $\mathrm{Tree}(P)$ of a preproof P is a tree of finite sequences of natural numbers such that each occurrence of a sequent or an inference rule receives a finite sequence. The root (empty sequence) ( ) is attached to the endsequent, and in an inference rule

$$\frac{a*(0,0):\Lambda_0 \quad \cdots \quad a*(0,n):\Lambda_n}{a:\Gamma}\ (r)\, a*(0)$$

where (r) is the name of the inference rule. Finite sequences are denoted by Roman letters $a, b, c, \ldots, I, J, K, \ldots$. Roman capitals $I, J, K, \ldots$ denote exclusively inference nodes. We will identify the attached sequence a with the occurrence of a sequent or an inference rule.

Let P be a preproof and $\gamma < \pi + \omega$ an o.d. in $Od(\Pi_N)$. For each sequent $a : \Gamma\, (a \in \mathrm{Tree}(P))$, we assign the *height* $\mathrm{h}_\gamma(a; P) < \pi + \omega$ of the node *a with the base height* γ in P as in [5] except we replace $\pi(\mathrm{T}_{22}) = \mu$ by $\pi(\mathrm{T}_N) = \pi$ and replace $\deg(A)$ by $\deg_N(A)$.

Then the *height* $\mathrm{h}(a; P)$ of a in P is defined to be the height with the base height $\gamma = 0$:
$$\mathrm{h}(a; P) := \mathrm{h}_0(a; P).$$

A pair (P, γ) of a preproof P and an o.d. γ is said to be *height regulated* if it enjoys the conditions in [5], or equivalently in [6], Definition 5.4. For the

rules $(\Sigma_i)^\sigma$, this requires the condition: If $a:\Gamma$ is the lowersequent of a rule $(\Sigma_i)^\sigma\, a*(0)\,(1\leq i\leq N)$ in P, then $\mathrm{h}_\gamma(a;P)\leq\sigma+i-2$ if $i=N-1,N$. Otherwise $\mathrm{h}_\gamma(a;P)\leq\sigma+i-1$.

Therefore for the uppersequent $a*(0,k):\Lambda$ of a $(\Sigma_i)^\sigma$ we have $\mathrm{h}_\gamma(a*(0,k);P)=\sigma+i-1$. Note that this implies that there is no nested rules $(\Sigma_i)^\sigma$, i.e., there is no $(\Sigma_i)^\sigma$ below any $(\Sigma_i)^\sigma$ for $i\geq N-1$.

A preproof is height regulated iff $(P,0)$ is height regulated.

Let P be a preproof and $\gamma<\pi+\omega$. Assume that (P,γ) is height regulated. Then the o.d $o_\gamma(a;P)\in O(\Pi_N)$ assigned to each node a in the underlying tree Tree(P) of P is defined exactly as in [6].

Furthermore for $\tau\in R\cap Od(\Pi_N)$, o.d.'s $B_{\tau,\gamma}(a;P), Bk_{\tau,\gamma}(a;P)\in O(\Pi_N)$ are assigned to each sequent node a such that $\mathrm{h}_\gamma(a;P)\leq\tau\in R$ as in [6]. Namely

$$
\begin{array}{rcl}
B_{\tau,\gamma}(a;P) & := & \left\{\begin{array}{l}
\pi\cdot o_\gamma(a;P) \\
\ \text{if } \mathrm{h}_\gamma(a;P)=\tau=\pi \\
\max\{\mathcal{B}_\pi(o_\gamma(a;P)),\mathcal{B}_{>\tau}(\{\tau\}\cup(a;P))\}+\omega^{o_\gamma(a;P)} \\
\ \text{if } \mathrm{h}_\gamma(a;P)<\pi
\end{array}\right. \\
Bk_{\tau,\gamma}(a;P) & := & B_{\tau,\gamma}(a;P)+\mathcal{K}_\tau(a;P)
\end{array}
$$

$B_\tau(a;P)$ $[Bk_\tau(a;P)]$ denotes $B_{\tau,0}(a;P)$ $[Bk_{\tau,0}(a;P)]$, resp.

Then propositions and lemmata (Rank Lemma 7.3, Inversion Lemma 7.9, etc.) in Section 9 of [5] and Replacement Lemma 5.15 in [6] hold also for T_{Nc}.

Lemma 4.2 yields $o_\gamma(a;P)\in Od(\Pi_N)$ for each node $a\in$ Tree(P) if (P,γ) is height regulated and $\gamma<\pi+\omega$.

Definition 5.2 Let $\mathcal{T}$ be a branch in a preproof P and J a rule $(\Sigma_i)^\sigma$.

1. *Left branch:* $\mathcal{T}$ is a *left branch* of J if
   - *(a)* $\mathcal{T}$ starts with a lowermost sequent Γ such that $h(\Gamma)\geq\pi$,
   - *(b)* each sequent in $\mathcal{T}$ contains an ancestor of the left cut formula of J and
   - *(c)* $\mathcal{T}$ ends with the left uppersequent of J.
2. *Right branch:* $\mathcal{T}$ is a *right branch* of J if
   - *(a)* $\mathcal{T}$ starts with a lowermost sequent Γ such that Γ is a lowersequent of a basic rule whose principal formula is an ancestor of the right cut formula of J and
   - *(b)* $\mathcal{T}$ ends with the right uppersequent of J.

Chains in a preproof are defined as in Definition 6.1 of [6] when we replace $((c\Pi_3),(\Sigma_3))$, $((c\Sigma_2),(\Sigma_2))$ by $((c\Pi_N),(\Sigma_N))$, $((c\Sigma_{N-1}),(\Sigma_{N-1}))$. For definitions related to chains such as *starting with*, *top*, *branch* of a chain, *passing through*, see Definition 6.1 of [6]. Also *rope sequence* of a rule, the *end* of a rope sequence and the *bar* of a rule are defined as in Definition 6.2 of [6]. Moreover a *chain analysis* for a preproof together with the *bottom* of a rule is defined as in Definition 6.3 of [6].

Definition 5.3 *Q part of a chain and the i-origin.*

1. Let $\mathcal{C} = J_0, J_0', \ldots, J_n, J_n'$ be a chain starting with a $(c)_\sigma\, J_n$. Put

   (a) $In(\mathcal{C}) := In(J_n) := In(\sigma)$.

   (b) $in_i(\mathcal{C}) := in_i(J_n) := in_i(\sigma)$ for $2 \le i < N$.

   (c) $st_i(\mathcal{C}) := st_i(J_n) := st_i(\sigma)$, $rg_i(\mathcal{C}) := rg_i(J_n) := rg_i(\sigma)$ where $st_i(\mathcal{C}) \uparrow \,\&\, rg_i(\mathcal{C}) \uparrow$ if $st_i(\sigma) \uparrow \,\&\, rg_i(\sigma) \uparrow$.

   (d) J_k is the *i-origin* of the chain $\mathcal{C}$ or the rule J_n if J_k is a rule $(c)^\kappa$ with $\kappa = rg_i(\sigma) \downarrow$.

   (e) J_k is the *i-predecessor* of J_n, denoted by $J_k = pd_i(J_n)$ or *i-predecessor* of the chain $\mathcal{C}$, denoted by $J_k = pd_i(\mathcal{C})$ if J_k is a rule $(c)_\rho$ with $\rho = pd_i(\sigma)$.

Definition 5.4 *Knot and rope.*
Assume that a chain analysis for a preproof P is given and by a chain we mean a chain in the chain analysis.

1. *i-knot:* Let K be a rule $(\Sigma_i)^\sigma$ $(1 \le i \le N-2)$. We say that K is an *i-knot* if there are an uppermost rule $(c)^\sigma\, J_{lw}$ below K and a chain $\mathcal{C}$ such that J_{lw} is a member of $\mathcal{C}$ and $\mathcal{C}$ passes through the left side of K.

   The rule J_{lw} is said to be the *lower rule* of the i-knot K. The member $(c)_\sigma\, J_{ul}$ of the chain $\mathcal{C}$ is the *upper left rule* of K and a rule $(c)_\sigma\, J_{ur}$ which is above the right uppersequent of K is an *upper right rule* of K if such a rule $(c)_\sigma\, J_{ur}$ exists.

$$
\begin{array}{c}
\vdots\ \mathcal{C} \qquad\qquad \vdots \\
\dfrac{\Gamma, \neg A^\sigma \qquad A^\sigma, \Lambda}{\Gamma, \Lambda}\ (\Sigma_i)^\sigma\ K \\
\vdots \\
\dfrac{\Delta}{\Delta'}\ \text{uppermost } (c)^\sigma\ J_{lw} \in \mathcal{C} \\
\vdots
\end{array}
$$

2. A rule is a *knot* if it is an i-knot for some $i > 1$.
   Remark. Note that a 1-knot (Σ_1) is not a knot by definition.

3. Let K be a knot, J_{lw} the lower rule of K and J_{ur} an upper right rule of K. Then we say that K is a *knot of J_{ur} and J_{lw}*.

4. Let $\mathcal{C}_n = J_0, \ldots, J_n$ be a chain starting with J_n and K a knot. K is a *knot for the chain $\mathcal{C}_n$* or *the rule J_n* if

   (a) the lower rule J_{lw} of K is a member $J_k\,(k < n)$ of $\mathcal{C}_n$,

   (b) $\mathcal{C}_n$ passes through the right side of K, and

(c) for any $k < n$ the chain $\mathcal{C}_k$ starting with J_k does not pass through the right side of K.

The knot K is a merging rule of the chain $\mathcal{C}_n$ and the chain $\mathcal{C}_k$ starting with the lower rule $J_{lw} = J_k$.

$$
\begin{array}{c}
\vdots\, \mathcal{C}_k \qquad \vdots\, \mathcal{C}_n \\
\dfrac{\Gamma, \neg A^\sigma \quad A^\sigma, \Lambda}{\Gamma, \Lambda}\ (\Sigma_i)^\sigma\, K \\
\vdots \\
\dfrac{\Delta_k}{\Delta'_k}\ \text{uppermost } (c)^\sigma\ J_{lw} = J_k \in \mathcal{C}_n \\
\vdots \\
\dfrac{\Delta_n}{\Delta'_n}\ J_n
\end{array}
$$

5. A series $\mathcal{R}_{J_0} = J_0, \ldots, J_{n-1}\, (n \geq 1)$ of rules (c) is said to be the *rope starting with* J_0 if there is an increasing sequence of numbers (uniquely determined)

$$0 \leq n_0 < n_1 < \cdots < n_l = n - 1\, (l \geq 0) \tag{11}$$

for which the following hold:

(a) each J_{n_m} is the bottom of $J_{n_{m-1}+1}$ for $m \leq l\, (n_{-1} = -1)$,

(b) there is an uppermost knot K_m such that J_{n_m} is an upper right rule and J_{n_m+1} is the lower rule of K_m for $m < l$, and

(c) there is no knot whose upper right rule is $J_{n_l} = J_{n-1}$.

We say that the rule J_{n-1} is the *edge of the rope* $\mathcal{R}_{J_0}$ or *the edge of the rule* J_0.
For a rope the increasing sequence of numbers (11) is called the *knotting numbers* of the rope.
Remark. These knots K_m are uniquely determined for a *proof* defined below.

6. Let K_{-1} be an i_{-1}-knot $(i_{-1} \geq 1)$ and J_0 the lower rule of K_{-1}. The *left rope* ${}_{K_{-1}}\mathcal{R}$ *of* K_{-1} is inductively defined as follows:

(a) Pick the lowermost rule $(c)\, J_{n_0}$ such that the chain $\mathcal{C}$ starting with J_{n_0} passes through the left side of the i_{-1}-knot K_{-1} and J_0 is a member of $\mathcal{C}$. Let ${}_0\mathcal{R} = I_0, \ldots, I_q$ be the part of the chain $\mathcal{C}$ with $J_0 = I_0 \,\&\, J_{n_0} = I_q$.

(b) If there exists an uppermost knot K_0 such that J_{n_0} is an upper right rule of K_0, then ${}_{K_{-1}}\mathcal{R}$ is defined to be a concatenation :

$${}_{K_{-1}}\mathcal{R} = {}_0\mathcal{R}^\frown {}_{K_0}\mathcal{R}$$

where ${}_{K_0}\mathcal{R}$ denotes the left rope of K_0.

(c) Otherwise. Set:

$$_{K_{-1}}\mathcal{R} = {}_0\mathcal{R}$$

Therefore for the left rope $_{K_{-1}}\mathcal{R} = J_0, \ldots, J_{n-1}$ of K_{-1} there exists a uniquely determined increasing sequence of numbers (11) such that:

(a) each J_{n_m} is the lowermost rule (c) such that the chain $\mathcal{C}$ starting with J_{n_m} passes through the left side of the i_{m-1}-knot K_{m-1} and $J_{n_{m-1}+1}$ is a member of $\mathcal{C}\,(n_{-1} = -1)$ for $m \le l$,

(b) there is an i_m-knot K_m $(i_m > 1)$ such that J_{n_m} is an upper right rule and J_{n_m+1} is the lower rule of K_m for $m < l$, and

(c) there is no knot whose upper right rule is $J_{n_l} = J_{n-1}$. (K_{-1} is the i_{-1}-knot whose lower rule is J_0.)

These numbers (11) is called the *knotting numbers* of the left rope and each knot $K_m\,(m < l)$ a *knot for the left rope.*

By the *left rope* $_{J_0}\mathcal{R}$ of the lower rule J_0 of K_{-1} we mean the left rope $_{K_{-1}}\mathcal{R}$ of K_{-1}.

When a rule $(\Sigma_{i+1})^{\sigma}\,K\,(0 < i < N)$ is resolved, we introduce a new rule $(\Sigma_i)^{\sigma_{n_{m(i+1)}+1}}$ at a sequent Φ, which is defined to be the *resolvent* of K and a $\sigma_{n_{m(i+1)}+1} \preceq \sigma$ defined as follows.

Definition 5.5 *Resolvent*
Let K be a rule $(\Sigma_{i+1})^{\sigma}\,(0 < i < N)$. The *resolvent* of the rule K is a sequent $a : \Phi$ defined as follows: let K' denote the lowermost rule $(\Sigma_{i+1})^{\sigma}$ below or equal to K and $b : \Psi$ the lowersequent of K'.

Case 1 The case when there exists an $(i+1)$-knot $(\Sigma_{i+1})^{\sigma}$ which is between an uppersequent of K and $b : \Psi$: Pick the uppermost such knot $(\Sigma_{i+1})^{\sigma}\,K_{-1}$ and let $_{K_{-1}}\mathcal{R} = J_0, \ldots, J_{n-1}$ denote the left rope of K_{-1}. Each J_p is a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$. Let

$$0 \le n_0 < n_1 < \cdots < n_l = n - 1\,(l \ge 0) \tag{11}$$

be the knotting numbers of the left rope $_{K_{-1}}\mathcal{R}$ and K_m an i_m-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}$ of J_{n_m} and J_{n_m+1} for $m < l$. Put

$$m(i+1) = \max\{m : 0 \le m \le l\,\&\,\forall p \in [0, m)(i + 1 \le i_p)\} \tag{12}$$

Then the resolvent $a : \Phi$ is defined to be the uppermost sequent $a : \Phi$ below $J_{n_{m(i+1)}}$ such that $h(a; P) < \sigma_{n_{m(i+1)}+1} + i$.

Case 2 Otherwise: Then the resolvent $a\Phi$ is defined to be the sequent $b : \Psi$.

Definition 5.6 Let J and J' be rules in a preproof such that both J and J' are one of rules $(\Sigma_i)\,(1 \le i \le N - 1)$ and J is above the right uppersequent of J'. We say that J *foreruns* J' if any right branch $\mathcal{T}$ of J' *is left to* J, i.e., there

exists a merging rule K such that $\mathcal{T}$ passes through the left side of K and the right uppersequent of K is equal to or below the right uppersequent of J.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_0,\neg A^{\sigma_0}\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}A^{\sigma_1}\\ \vdots \text{ right branch } \mathcal{T} \text{of } J'\\ \Gamma_1,\neg B\end{array} \qquad \begin{array}{c}\dfrac{\Gamma_2,\neg C \quad C,\Lambda_2}{\Gamma_2,\Lambda_2}\,(\Sigma_j)\,J\\ \vdots\\ B,\Lambda_1\end{array}}{\Gamma_1,\Lambda_1}\,K \\ \vdots \\ A^{\sigma_0},\Lambda_0\end{array}}{\Gamma_0,\Lambda_0}\,(\Sigma_i)^{\sigma_0}\,J'$$

If J foreruns J', then resolving steps of J precede ones of J'. In other words we have to resolve J in advance in order to resolve J'.

Definition 5.7 Let $\mathcal{R} = J_0, \ldots, J_{n-1}$ denote a series of rules (c). Each J_p is a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$Assume that J_0 is above a rule $(\Sigma_i)^\sigma\, I$ and $\sigma = \sigma_p$ for some p with $0 < p \leq n$. Then we say that the series $\mathcal{R}$ *reaches to* the rule I.

In a *proof* defined in the next definition, if a series $\mathcal{R} = J_0, \ldots, J_{n-1}$ reaches to the rule $(\Sigma_i)^\sigma\, I$, then either $\mathcal{R}$ passes through I in case $p < n$, or the subscript σ_n of the last rule $(c)^{\sigma_{n-1}}_{\sigma_n}\, J_{n-1}$ is equal to σ, i.e., J_{n-1} is a lowermost rule (c) above I.

Definition 5.8 *Proof*
Let P be a preproof. Assume a chain analysis for P is given. The preproof P *together with the chain analysis* is said to be a *proof* in T_{Nc} if it satisfies the following conditions besides the conditions **(pure)**, **(h-reg)**, **(c:side)**, **(c:bound)**, **(next)**, **(h:bound)**, **(ch:pass)** (a chain passes through only rules $(c), (h), (\Sigma_i)\,(i < N)$), **(ch:left)**, which are the same as in [6]:

(st:bound) Let $\mathcal{C}$ be a chain, $i \in In(\mathcal{C})$ and $a : \Gamma$ be the uppersequent of the i-origin of the chain $\mathcal{C}$.

(st:bound1) Let $i = N - 1$. Then

$$o(a; P) \leq st_{N-1}(\mathcal{C}).$$

(st:bound2) Let $i < N - 1$ and $\kappa = rg_i(\mathcal{C})$. Then for an α

$$st_i(\mathcal{C}) = d_{\kappa^+}\alpha$$

and

$$B_\kappa(a; P) \leq \alpha.$$

(ch:link) *Linking chains:* Let $\mathcal{C} = J_0, J_0', \ldots, J_n, J_n'$ and $\mathcal{D} = I_0, I_0', \ldots, I_m, I_m'$ be chains such that J_i is a rule $(c)^{\tau_i}_{\tau_{i+1}}$ and I_i a rule $(c)^{\sigma_i}_{\sigma_{i+1}}$. Assume that branchs of these chains intersect. Then one of the following three types must occur (Cf. [6] for **Type1 (segment)** and **Type2 (jump)**):

Type1 (segment) : One is a part of the other, i.e.,

$$n \leq m \,\&\, J_i = I_i$$

or vice versa.

Assume that there exists a merging rule K such that $\mathcal{C}$ passes through the left side of K and $\mathcal{D}$ the right side of K. Then by **(ch:left)** the merging rule K is a $(\Sigma_l)^{\tau_j}$ for some $j \leq n$ and some l with $1 \leq l \leq N-2$.

Type2 (jump) : The case when there is an $i \leq m$ so that

1. J'_{j-1} is above K and J_j is below K,
2. I_i is above K,
3. I'_i is below J'_n and
4. $\sigma_{i+1} < \tau_{n+1}$.

Type3 (merge) : The case when $\tau_j = \sigma_j$. Then it must be the case:

1. $l > 1$,
2. I'_{j-1} and J'_{j-1} are rules $(\Sigma_{N-1})^{\tau_j}$ above K, and
3. $n < m \,\&\, J_{j+k} = I_{j+k} \,\&\, J'_{j+k} = I'_{j+k}$ for any k with $j \leq j+k \leq n$.

That is to say, $\mathcal{C}$ and $\mathcal{D}$ share the part from $J_j = I_j$ to $J_n = I_n$, the right chain $\mathcal{D}$ has to be longer $n < m$ than the left chain $\mathcal{C}$ and the merging rule K is not a rule (Σ_1).

If **Type2 (jump)** or **Type3 (merge)** occurs for chains $\mathcal{C}$ and $\mathcal{D}$, then we say that $\mathcal{D}$ *foreruns* $\mathcal{C}$, since the resolving of the chain $\mathcal{D}$ precedes the

resolving of the chain $\mathcal{C}$.

$$
\dfrac{\dfrac{\dfrac{\overset{\vdots}{\Phi_{j-1},\neg A^{\tau_j}_{j-1}} \quad \overset{\vdots}{A^{\tau_j}_{j-1},\Psi_{j-1}}}{\Phi_{j-1},\Psi_{j-1}}\,(\Sigma_{N-1})^{\tau_j}\,J'_{j-1}}{\overset{\vdots\,\mathcal{C}}{\Phi,\neg A^{\tau_j}}} \qquad \dfrac{\dfrac{\overset{\vdots}{\Pi,\neg B^{\sigma_j}} \quad \overset{\vdots}{B^{\sigma_j},\Delta}}{\Pi,\Delta}\,(\Sigma_{N-1})^{\sigma_j}\,I'_{j-1}}{\overset{\vdots\,\mathcal{D}}{A^{\tau_j},\Psi}}}{\Phi,\Psi}\,(\Sigma_l)^{\tau_j}\,K
$$

$$
\begin{array}{c}
\Phi,\Psi\\
\vdots\\
\dfrac{\Gamma_j}{\Gamma'_j}\ (c\Sigma_{N-1})^{\tau_j}_{\tau_{j+1}}\,J_j = (c\Sigma_{N-1})^{\sigma_j}_{\sigma_{j+1}}\,I_j\\
\vdots\\
\dfrac{\Gamma_n}{\Gamma'_n}\ (c\Sigma_{N-1})^{\tau_n}_{\tau_{n+1}}\,J_n = (c\Sigma_{N-1})^{\sigma_n}_{\sigma_{n+1}}\,I_n\\
\vdots\\
\dfrac{\overset{\vdots}{\Phi_n,\neg A^{\tau_{n+1}}_n} \quad \overset{\vdots}{A^{\tau_{n+1}}_n,\Psi_n}}{\Phi_n,\Psi_n}\ (\Sigma_{N-1})^{\tau_{n+1}}\,J'_n = (\Sigma_{N-1})^{\sigma_{n+1}}\,I'_n\\
\vdots\\
\dfrac{\Gamma_m}{\Gamma'_m}\ (c)^{\sigma_m}_{\sigma_{m+1}}\,I'_m
\end{array}
\qquad Type3
$$

(ch:Qpt) Let $\mathcal{C} = J_0, \dots, J_n$ be a chain with a $(c)^{\sigma_p}_{\sigma_{p+1}}\,J_p\,(p \leq n)$ and put $\rho = \sigma_{n+1}$. Then by **(ch:link)** there exists a uniquely detremined increasing sequence of numbers

$$0 \leq n_0 < n_1 < \cdots < n_l = n-1\,(l \geq 0) \tag{11}$$

such that for each $m < l$ there exists an i_m-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}\,K_m\,(2 \leq i_m \leq N-2)$ for the chain $\mathcal{C}$. (The i_m-knot K_m is the merging rule of the chain $\mathcal{C}$ and the chain starting with the rule J_{n_m+1}, cf. **Type3 (merge)**.) These numbers are called the *knotting numbers* of the chain $\mathcal{C}$.

Then $pd_i(\rho), In(\rho), rg_i(\rho)$ have to be determined as follows:

1. For $2 \leq i < N$,

$$pd_i(\rho) = \sigma_{n_{m(i)}+1}$$

with

$$m(i) = \max\{m : 0 \leq m \leq l \,\&\, \forall p \in [0,m)(i \leq i_p)\} \tag{13}$$

that is to say,

$$J_{n_{m(i)}} = pd_i(J_n).$$

2. For $2 \leq i < N-1$

$$
\begin{array}{rcl}
i \in In(\mathcal{C}) = In(\rho) & \Leftrightarrow & \exists p \in [0, m(i))(i_p = i)\\
& \Leftrightarrow & \exists p \in [0,l)(i_p = i \,\&\, \forall q < p(i_q > i))\\
& \Leftrightarrow & m(i) > m(i+1) = \min\{m < l : i_m = i\}
\end{array}
$$

And by the definition $N-1 \in In(\mathcal{C}) = In(\rho)$.

3. For $i \in In(\mathcal{C})\,\&\, i \neq N-1$,

(a) The case when there exists a q such that

$$\exists p[n_{m(i)} \geq p \geq q > n_{m(i+1)} \,\&\, \rho \prec_i \sigma_{p+1} \,\&\, \sigma_q = rg_i(\sigma_{p+1})] \quad (14)$$

Then

$$rg_i(\rho) = \sigma_q$$

where q denotes the minimal q satisfying (14).

(b) Otherwise.

$$rg_i(\rho) = pd_i(\rho) = \sigma_{n_{m(i)}+1}$$

(lbranch) Any left branch of a (Σ_i) is the rightmost one in the left upper part of the (Σ_i).

(forerun) Let J^{lw} be a rule $(\Sigma_j)^\sigma$. Let $\mathcal{R}_{J_0} = J_0, \ldots, J_{n-1}$ denote the rope starting with a $(c)\,J_0$. Assume that J_0 is above the right uppersequent of J^{lw} and the series $\mathcal{R}_{J_0}$ reaches to the rule J^{lw}. Then there is no merging rule K, *cf.* the figure below, such that

1. the chain $\mathcal{C}_0$ starting with J_0 passes through the right side of K, and

2. a right branch $\mathcal{T}$ of J^{lw} passes through the left side of K.

$$
\begin{array}{c}
\vdots\, \mathcal{T} \qquad \vdots\, \mathcal{C}_0 \\
\dfrac{\Gamma, \neg B \quad\quad B, \Lambda}{\Gamma, \Lambda}\; K \\
\vdots \\
\dfrac{\Gamma_0}{\Gamma'_0}\; (c)\, J_0 \\
\vdots\, \mathcal{R}_{J_0} \\
\dfrac{\Phi, \neg A \qquad\qquad A, \Psi}{\Phi, \Psi}\; (\Sigma_j)^\sigma\, J^{lw}
\end{array}
$$

(uplw) Let J^{lw} be a rule $(\Sigma_j)^\sigma$ and J^{up} an i-knot $(\Sigma_i)^{\sigma_0}$ $(1 \leq i, j \leq N)$. Let J_0 denote the lower rule of J^{up}. Assume that the left rope ${}_{J^{up}}\mathcal{R} = J_0, \ldots, J_{n-1}$ of J^{up} reaches to the rule J^{lw}. Then

(uplwl) if J^{up} is above the left uppersequent of J^{lw}, then $j < i < N$.

$$
\begin{array}{c}
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ \mathcal{C}_0\\ \Gamma,\neg B\end{array}\qquad\begin{array}{c}\vdots\\ B,\Lambda\end{array}}{\Gamma,\Lambda}\ (\Sigma_i)^{\sigma_0}\ J^{up}\\
\vdots\\
\dfrac{\Gamma_0}{\Gamma'_0}\ (c)^{\sigma_0}\ J_0\\
\vdots\ {}_{J^{up}}\mathcal{R}\\
\Phi,\neg A
\end{array}
\qquad
\begin{array}{c}\vdots\\ A,\Psi\end{array}
}{\Phi,\Psi}\ (\Sigma_j)^{\sigma}\ J^{lw}
\end{array}
\Longrightarrow j < i
$$

where $\mathcal{C}_0$ denotes the chain starting with J_0, and

(uplwr) if J^{up} is above the right uppersequent of J^{lw} and $i \leq j \leq N$, then the rule $(\Sigma_i)^{\sigma_0}\, J^{up}$ foreruns the rule $(\Sigma_j)^{\sigma}\, J^{lw}$, *cf.* Proposition 2.3 in Subsection 2.2.

In other words if there exists a right branch $\mathcal{T}$ of J^{lw} as shown in the following figure, then $j < i$.

$$
\dfrac{
\begin{array}{c}\vdots\\ \Phi,\neg A\end{array}
\qquad
\begin{array}{c}
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ \mathcal{C}_0\\ \Gamma,\neg B\end{array}\qquad\begin{array}{c}\vdots\ \mathcal{T}\\ B,\Lambda\end{array}}{\Gamma,\Lambda}\ (\Sigma_i)^{\sigma_0}\ J^{up}\\
\vdots\\
\dfrac{\Gamma_0}{\Gamma'_0}\ (c)^{\sigma_0}\ J_0\\
\vdots\\
\Pi,\neg C
\end{array}
\qquad
\begin{array}{c}\vdots\ \mathcal{T}\\ C,\Delta\end{array}
}{\Pi,\Delta}\ \exists K\\
\vdots\ {}_{J^{up}}\mathcal{R}\\
A,\Psi
\end{array}
}{\Phi,\Psi}\ (\Sigma_j)^{\sigma}\ J^{lw}
$$

Decipherment. These provisos for a preproof to be a proof are obtained by inspection to rewrited proof figures. We decipher only additional provisos from [6].

(ch:link) Now a new type of linking chains, **Type3 (merge)** enters, cf. Subsection 2.1.

For a chain $\mathcal{D} = I_0, I'_0, \ldots, I_m, I'_m$ and a member $I_n\,(n < m)$ of $\mathcal{D}$ let $\mathcal{C} = J_0, J'_0, \ldots, J_n, J'_n$ denote the chain starting with $J_n = I_n$. Then there are two possibilities:

Type1 (segment) $\mathcal{C}$ is a part $I_0, \ldots, I_n$ of $\mathcal{D}$ and hence the tops I_0 and J_0 are identical.

Type3 (merge) The branch of $\mathcal{C}$ is left to the branch of $\mathcal{D}$.

(st:bound), (ch:Qpt) By these provisos we see that an o.d. ρ is in $Od(\Pi_N)$ for a newly introduced rule $(c)_\rho$, cf. Propositions 2.2, 2.3 in Subsection 2.2, Lemma 5.8 below and the case **M5.2** in the next Section 6.

(uplwl) By the proviso we see that a preproof P' which is resulted from a proof P is again a proof with respect to the proviso **(ch:Qpt)**, cf. Lemma 5.7.2.

(uplwr), (forerun), (lbranch) By these provisos we see that a preproof P' which is resulted from a proof P by resolving a rule (Σ_{i+1}) is again a proof with respect to the provisos **(forerun)** and **(uplw)**, cf. Proposition 2.3 in Subsection 2.2, the case **M7.2** in the next Section 6, Lemma 5.5 and Lemma 5.4.

In the following any sequent and any rule are in a fixed proof.

As in the previous paper [6] we have the following lemmata. Lemma 5.1 follows from the provisos **(h-reg)** and **(ch:link)** in Definition 5.8, , Lemma 5.2 from **(h-reg)** and **(c:bound1)** and Lemma 5.3 from **(h-reg)**.

Lemma 5.1 *Let J be a rule $(c)_\sigma$ and J' the trace $(\Sigma_{N-1})^\sigma$ of J. Let J_1 be a rule $(c)^\sigma$ below J'. If there exists a chain $\mathcal{C}$ to which both J and J_1 belong, then J_1 is the uppermost rule $(c)^\sigma$ below J and there is no rule (c) between J' and J_1.*

Lemma 5.2 *Let J_{top} be a rule $(c)^\pi$. Let Φ denote the bar of J_{top}. Assume that the branch $\mathcal{T}$ from J_{top} to Φ is the rightmost one in the upper part of Φ. Then no chain passes through Φ.*

Lemma 5.3 *Let J be a rule (c) and $b : \Phi$ the bar of the rule J. Then there is no (cut) I with $b \subset I \subset J$ nor a right uppersequent of a (Σ_N) I with $b \subset I * (1) \subseteq J$ between J and $b : \Phi$.*

The following lemma is used to show that a preproof P' which results from a proof P by resolving a rule (Σ_j) J^{lw} is again a proof with respect to the proviso **(uplwl)**, cf. the Claim 6.6 in the case **M7** in the next subsection.

Lemma 5.4 *Let J^{lw} be a rule (Σ_j). Assume that there exists a right branch $\mathcal{T}$ of J^{lw} such that $\mathcal{T}$ is the rightmost one in the upper part of J^{lw}. Then there is no i-knot (Σ_i) J^{up} above the right uppersequent of J^{lw} such that $i \leq j$ and the left rope ${}_{J^{up}}\mathcal{R}$ of J^{up} and J^{up} reaches to J^{lw}.*

Proof. Suppose such a rule J^{up} exists. By **(uplwr)** the rule J^{up} foreruns J^{lw}. Thus the branch $\mathcal{T}$ would not be the rightmost one.

$$
\begin{array}{c}
\vdots\ \mathcal{T} \\
\dfrac{\Psi, \neg B \quad B, \Phi}{\Psi, \Phi}\ (\Sigma_i)\, J^{up} \\
\vdots \\
\dfrac{\Gamma, \neg A \qquad A, \Lambda}{\Gamma, \Lambda}\ (\Sigma_j)\, J^{lw}
\end{array}
$$

$\square$

The following lemma is used to show that a preproof P' which results from a proof P by resolving a (Σ_{i+1}) is a proof with respect to the proviso **(uplwr)**, and to show a newly introduced rule (Σ_i) in such a P' does not split any chain, cf. the Claim 6.6 in the case **M7**.

Lemma 5.5 *Let J be a rule $(\Sigma_{i+1})^{\sigma_0}$ $(0 < i < N)$ and $b : \Phi$ the resolvent of J. Assume that the branch $\mathcal{T}$ from J to b is the rightmost one in the upper part of b. Then every chain passing through b passes through the right side of J.*

Proof. Let $a * (0)$ denote the lowermost rule $(\Sigma_{i+1})^{\sigma_0}$ below or equal to J, and $a : \Psi$ the lowersequent of $a * (0)$. The sequent $a : \Psi$ is the uppermost sequent below J such that $h(a; P) < \sigma_0 + i$ by **(h-reg)**.
Case 2. $b = a$: If a chain passes through a and a left side of a $(\Sigma_{i+1})^{\sigma_0}$ K_{-1} with $a \subset K_{-1} \subseteq J$, then the chain would produce an $(i+1)$-knot K_{-1}.
Case 1. Otherwise: Then there exists an $(i+1)$-knot $(\Sigma_{i+1})^{\sigma_0}$ with $a \subset K_{-1} \subseteq J$. Let $(\Sigma_{i+1})^{\sigma}$ K_{-1} denote the uppermost such knot and ${}_{K_{-1}}\mathcal{R} = J_0, \ldots, J_{n-1}$ the left rope of K_{-1}. Each J_p is a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$. Let

$$0 \leq n_0 < n_1 < \cdots < n_l = n - 1 \, (l \geq 0) \tag{11}$$

be the knotting numbers of the left rope ${}_{K_{-1}}\mathcal{R}$ and K_m an i_m-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}$ of J_{n_m} and J_{n_m+1} for $m < l$. Put

$$m(i+1) = \max\{m : 0 \leq m \leq l \,\&\, \forall p \in [0, m)(i + 1 \leq i_p)\} \tag{12}$$

Then the resolvent $b : \Phi$ is the uppermost sequent $b : \Phi$ below $J_{n_{m(i+1)}}$ such that

$$h(b; P) < \sigma_{n_{m(i+1)}+1} + i.$$

Put

$$m = m(+1), \sigma = \sigma_{n_m+1}.$$

Assume that there is a chain $\mathcal{C}$ passing through b. As in **Case 2** it suffices to show that the chain $\mathcal{C}$ passes through the right side of K_{-1}. Assume that this is not the case. Let $(c)^{\rho}_{\rho'}$ K denote the lowermost member of $\mathcal{C}$ which is above b.

Claim 5.1 *K is on the branch $\mathcal{T}$.*

Proof of the Claim 5.1. Assume that this is not the case. Then we see that there exists a merging rule $(\Sigma_j)^{\rho'}$ I and a member $(c)^{\rho'}$ K' of $\mathcal{C}$ such that the chain $\mathcal{C}$ passes through the left side of I. $K' \subset b \subset I$ and hence $h(K'; P) = \rho' \leq \sigma$. We see $\rho' = \sigma$ from **(h-reg)**.

Suppose $m = l$. Then by the definition of the left rope ${}_{K_{-1}}\mathcal{R}$, the rule $(\Sigma_j)^{\rho'}$ I is not a knot, i.e., $j = 1$. But then $h(I * (1); P) = h(K'; P) = \sigma$, and hence $I \subset b$. A contradiction. Therefore $m < l$ and $i_m \leq i$. This means $K_m * (1) \subseteq b \subset I$. On the other hand we have $1 \leq i < j$ by $b \subset I$, and by Lemma 5.1 K' is the uppermost rule $(c)^{\sigma}$ below $(\Sigma_j)^{\sigma}$ I. Therefore $(\Sigma_j)^{\rho'}$ I would be a

knot below J_{n_m}. On the other side K_m is the uppermost knot below J_{n_m}. This is a contradiction.

$$
\begin{array}{c}
\dfrac{\Pi}{\Pi'}\ (c)^{\rho}_{\sigma}\,K\in\mathcal{C} \qquad\qquad J \\
\vdots\ \mathcal{C} \qquad\qquad\qquad\quad \vdots \\
\dfrac{\Gamma_0,\neg A_0 \qquad\qquad\qquad A_0,\Lambda_0}{\Gamma_0,\Lambda_0}\ (\Sigma_j)^{\sigma}\,I \\
\vdots \\
b:\Phi \\
\vdots \\
\dfrac{\Gamma_1,\neg A_1 \quad A_1,\Lambda_1}{\Gamma_1,\Lambda_1}\ (\Sigma_{i_m})^{\sigma}\,K_m \\
\vdots \\
\dfrac{\Delta}{\Delta'}\ (c)^{\sigma}\,K'\in\mathcal{C}
\end{array}
$$

□

Then as in the proof of Lemma 7.13 of [6] we see that $K = J_{n_m}$, i.e., $(c)^{\rho}_{\rho'}\,K$ and $(c)^{\sigma_{n_m}}_{\sigma}\,J_{n_m}$ coincide. Consider the chain $\mathcal{C}_m$ starting with $(c)^{\sigma_{n_m}}_{\sigma}\,J_{n_m}$. Then by **(ch:link)** either $\mathcal{C}_m$ is a segment of $\mathcal{C}$ by **Type1(segment)**, or $\mathcal{C}$ foreruns $\mathcal{C}_m$ by **Type3(merge)**. Since $(c)^{\sigma_{n_m}}_{\sigma}\,J_{n_m}$ is the lowest one such that $\mathcal{C}_m$ passes through the left side of K_{m-1} and $J_{n_{m-1}+1}$ is a member of $\mathcal{C}_m$, **Type1(segment)** does not occur. In **Type3(merge)** K_{m-1} has to be the merging rule of $\mathcal{C}_m$ and $\mathcal{C}$ since, again, $(c)^{\sigma_{n_m}}_{\sigma}\,J_{n_m}$ is the lowest one, and the branch $\mathcal{T}$ is the rightmost one. Therefore $\mathcal{C}$ passes through the right side of K_{m-1}. If $m = 0$, then we are done. Otherwise we see the chain $\mathcal{C}$ and the chain $\mathcal{C}_{m-1}$ starting with $(c)^{\sigma_{n_{m-1}}}_{\sigma}\,J_{n_{m-1}}$ has to share the rule $(c)^{\sigma_{n_{m-1}}}_{\sigma}\,J_{n_{m-1}}$. As above we see that $\mathcal{C}$ passes through the right side of K_{m-2}, and so forth. □

Lemma 5.6 *Let $\mathcal{C} = J_0, \dots, J_n$ be a chain with rules $(c)^{\sigma_p}_{\sigma_{p+1}}\,J_p$ for $p \leq n$, and $(\Sigma_j)^{\sigma_p}\,K\,(p < n)$ a rule such that $\mathcal{C}$ passes through the right side of K and the chain $\mathcal{C}_p$ stating with J_p passes through the left side of K. Further let $\mathcal{R} = {}_K\mathcal{R} = J_p, \dots, J_{q-1}\,(q \leq n)$ denote the left rope of the j-knot K. Then the chain $\mathcal{C}_q$ starting with J_q is a part of the chain $\mathcal{C} = \mathcal{C}_n$, $\mathcal{C}_q \subseteq \mathcal{C}$. Therefore any knot for the chain $\mathcal{C}$ is below J_q and $q < n$, and in particular, if K is a knot for the chain $\mathcal{C}$, then $b = n$, cf. Definition 5.4.3.*

Proof. Suppose $q < n$. By the Definiton 5.4.6 there is no knot of J_{q-1} and J_q. Let I_q denote a knot such that the chain $\mathcal{C}_{q-1}$ starting with J_{q-1} passes through the left side of I_q. $c \subseteq b$. From the definition of a left rope we see that the chain $\mathcal{C}_q$ starting with J_q does not pass through the left side of the knot I_q. Therefore by **(ch:link) Type1 (segment)** the chain $\mathcal{C}_q$ must be a part of the

chain $\mathcal{C}$, $\mathcal{C}_q \subset \mathcal{C}$, i.e., the top of the chain $\mathcal{C}_q$ is the top J_0 of $\mathcal{C}$.

$$
\begin{array}{c}
\vdots\, \mathcal{C}_p \qquad \vdots\, \mathcal{C}_q \subset \mathcal{C} \\
\dfrac{\Phi_p, \neg A_p \quad A_p, \Psi_p}{\Phi_p, \Psi_p}\, (\Sigma_j)^{\sigma_p}\, K \\
\vdots \\
\dfrac{\Gamma_p}{\Gamma'_p}\, (c)^{\sigma_p}\, J_p \\
\vdots\, \mathcal{C}_{q-1} \qquad\qquad \vdots\, \mathcal{R} \\
\dfrac{\Phi_{q-1}, \neg A_{q-1} \quad A_{q-1}, \Psi_{q-1}}{\Phi_{q-1}, \Psi_{q-1}}\, I_q \\
\vdots\, \mathcal{R} \\
\dfrac{\Gamma_{q-1}}{\Gamma'_{q-1}}\, (c)_{\sigma_q}\, J_{q-1} \\
\vdots\, \mathcal{C}_q \\
\dfrac{\Gamma_q}{\Gamma'_q}\, (c)^{\sigma_q}\, J_q \\
\vdots\, \mathcal{C} \\
\dfrac{\Gamma_n}{\Gamma'_n}\, J_n
\end{array}
$$

$\Box$

The following Lemma 5.7 is a preparation for Lemma 5.8. From the Lemma 5.8 we see that an o.d. ρ is in $Od(\Pi_N)$ for a newly introduced rule $(c)_\rho$, cf. the case **M5.2** in the next subsection.

In the following Lemma 5.7, J denotes a rule $(c)_\rho$ and $\mathcal{C} = J_0, \ldots, J_n$ the chain starting with $J_n = J$. Each J_p is a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$ for $p \leq n$ with $\sigma_{n+1} = \rho$.

K denotes a rule $(\Sigma_j)^{\sigma_a}$ $(j \leq N-2, 0 < a \leq n)$ such that the chain $\mathcal{C}$ passes through K. If $\mathcal{C}$ passes through the left side of K, then $j \leq N-2$ holds by **(ch:left)**.

J_{a-1} denotes the lowermost member $(c)_{\sigma_a}$ of $\mathcal{C}$ above K, $K \subset J_{a-1}$.

Let

$$0 \leq n_0 < n_1 < \cdots < n_l = n-1\, (l \geq 0) \tag{11}$$

be the knotting numbers of the chain $\mathcal{C}$, cf. **(ch:Qpt)**, and K_m an i_m-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}$ of J_{n_m} and J_{n_m+1} for $m < l$. Let $m(i)$ denote the number

$$m(i) = \max\{m : 0 \leq m \leq l \,\&\, \forall p \in [0, m)(i \leq i_p)\} \tag{13}$$

Lemma 5.7 *(cf. Proposition 2.2 in Subsection 2.2.)*

1. *Let $m \leq m(i)$. Then*

$$i \leq i_{m-1} \,\&\, \sigma_{n_m+1} \prec_i \sigma_{n_{m-1}+1}.$$

2. *Assume that $\mathcal{C}$ passes through the left side of the rule K, i.e., $K * (0) \subset J_{a-1}$. Then J_{a-1} is the upper left rule of K. Let $i \leq j$.*

   (a) *$\rho \prec_i \sigma_a$,*
   and hence

   (b) *the i-predecessor of J is equal to or below J_{a-1}, and*

   (c) *if K_p is an i_p-knot (Σ_{i_p}) for the chain $\mathcal{C}$ above K, then $j < i_p$.*

$$
\begin{array}{l}
\dfrac{\Phi_p, \neg A_p \quad A_p, \Psi_p \ \vdots\,\mathcal{C}}{\Phi_p, \Psi_p}\ (\Sigma_{i_p})^{\sigma_{n_p+1}}\, K_p \\
\qquad \vdots \\
\dfrac{\Gamma_{a-1}}{\Gamma'_{a-1}}\ (c)_{\sigma_a}\ J_{a-1} \\
\qquad \vdots\,\mathcal{C} \qquad\qquad\qquad \vdots \\
\dfrac{\Phi, \neg A \qquad\qquad\qquad A, \Psi}{\Phi, \Psi}\ (\Sigma_j)^{\sigma_a}\ K \\
\qquad\qquad \vdots \\
\qquad\qquad \dfrac{\Gamma_n}{\Gamma'_n}\ (c)^{\sigma_n}_{\rho}\ J_n = J \qquad\qquad \Longrightarrow \rho \prec_j \sigma_a \,\&\, j < i_p
\end{array}
$$

3. *Let J_{b-1} be a member of $\mathcal{C}$ such that $\rho \prec_i \sigma_b$ for an i with $2 \leq i \leq N-2$. Let $\mathcal{C}_{b-1}$ denote the chain starting with J_{b-1}. Assume that the chain $\mathcal{C}_{b-1}$ intersects $\mathcal{C}$ of* **Type3 (merge)** *in* **(ch:link)** *and $(\Sigma_j)\, K$ is the merging rule of $\mathcal{C}_{b-1}$ and $\mathcal{C}$.*

   Then $i \leq j$.

$$
\begin{array}{l}
\quad \vdots\,\mathcal{C}_{b-1} \qquad \vdots\,\mathcal{C} \\
\dfrac{\Phi, \neg A \qquad A, \Psi}{\Phi, \Psi}\ (\Sigma_j)\, K \\
\qquad \vdots \\
\qquad \dfrac{\Gamma_{b-1}}{\Gamma'_{b-1}}\ (c)_{\sigma_b}\ J_{b-1} \\
\qquad \vdots\,\mathcal{C} \\
\qquad \dfrac{\Gamma_n}{\Gamma'_n}\ (c)_{\sigma_{n+1}}\ J_n \qquad\qquad \&\, \sigma_{n+1} \prec_i \sigma_b \Longrightarrow i \leq j
\end{array}
$$

4. *Assume that $\mathcal{C}$ passes through the left side of the rule K, i.e., $K * (0) \subset J_{a-1}$. Let $i \leq j$.*

   Assume that the i-origin J_q of $\mathcal{C}$ is not below K, i.e., $\sigma_q = rg_i(\rho) \downarrow \Rightarrow q < a$. Then

   $$\forall b \in (a, n+1]\{\rho \preceq_i \sigma_b \prec_i \sigma_a \to i \notin In(\sigma_b)\}$$

   and hence

   $$\forall b \in (a, n+1]\{\rho \preceq_i \sigma_b \prec_i \sigma_a \to in_i(J) = in_i(J_{b-1}) = in_i(J_{a-1}) \text{ , i.e.,} \\ in_i(\rho) = in_i(\sigma_b) = in_i(\sigma_a)\}$$

   In particular by Lemma 5.7.2 we have

   $$\rho \prec_i \sigma_a \,\&\, in_i(J) = in_i(J_{a-1}) \text{ , i.e., } in_i(\rho) = in_i(\sigma_a).$$

5. *Assume that $\mathcal{C}$ passes through the left side of the rule K. Let J_{b-1} be a member of $\mathcal{C}$ such that J_{b-1} is below K, i.e., $a < b$, and assume that $\sigma_{n+1} \preceq_i \sigma := \sigma_b$ for an $i \leq j$. If $\sigma_q = rg_i(\sigma) \downarrow \Rightarrow q < a$, then*

   $$\forall d \in (a, b]\{\sigma \preceq_i \sigma_d \prec_i \sigma_a \to i \notin In(\sigma_d)\}$$

   and

   $$\sigma \prec_i \sigma_a.$$

   Hence

   $$\forall d \in (a, b]\{\sigma \preceq_i \sigma_d \prec_i \sigma_a \to in_i(\sigma_d) = in_i(\sigma_a)\} \,\&\, in_i(\sigma) = in_i(\sigma_a).$$

   The following figure depicts the case $\sigma_q = rg_i(\sigma) \downarrow$:

   $$\begin{array}{c}
\vdots\ \mathcal{C} \qquad\qquad\qquad\qquad \\
\dfrac{\Gamma_q}{\Gamma'_q}\ (c)^{rg_i(\sigma)}\ J_q \qquad\qquad \\
\vdots\ \mathcal{C} \qquad\qquad\qquad \vdots \\
\dfrac{\Phi, \neg A \qquad\qquad\qquad A, \Psi}{\Phi, \Psi}\ (\Sigma_j)^{\sigma_a}\ K \\
\vdots \\
\dfrac{\Gamma_{b-1}}{\Gamma'_{b-1}}\ (c)_\sigma\ J_{b-1} \\
\vdots\ \mathcal{C} \\
\dfrac{\Gamma_n}{\Gamma'_n}\ (c)_{\sigma_{n+1}}\ J_n
\end{array}$$

6. *Assume that the chain $\mathcal{C}$ passes through the left side of the rule K. For an $i \leq j$ assume that there exists a q such that*

   $$\exists p[n \geq p \geq q \geq a \,\&\, \rho \preceq_i \sigma_{p+1} \,\&\, \sigma_q = rg_i(\sigma_{p+1})].$$

   Pick the minimal such q_0 and put $\kappa = \sigma_{q_0}$. Then

(a) $\forall d \in (a, q_0]\{\sigma_{q_0} \preceq_i \sigma_d \prec_i \sigma_a \to i \not\in In(\sigma_d)\}$ *and* $in_i(J_{a-1}) = in_i(J_{q_0-1})$, *i.e.,* $in_i(\sigma_a) = in_i(\kappa)$ *and* $\kappa \preceq_i \sigma_a$.

(b) $\forall t[\rho \preceq_i \sigma_t \prec_i \kappa \Rightarrow rg_i(\sigma_t) \preceq_i \kappa]$.

7. *Assume that* $\mathcal{C}$ *passes through the left side of the rule* K. *Let* J_{b-1} *be a member of* $\mathcal{C}$ *such that* J_{b-1} *is below* K, *i.e.,* $a < b$ *and* $\rho \preceq_i \sigma := \sigma_b$ *for an* $i \leq j$. *Suppose* $rg_i(\sigma) \downarrow$ *and put* $\sigma_q = rg_i(\sigma)$. *If the member* $(c)^{\sigma_q} J_q$ *is below* K, *i.e.,* $a \leq q$, *then for* $st_i(\sigma) = d_{\sigma_q^+}\alpha$, *cf.* **(st:bound)**,

$$B_{\sigma_q}(c; P) \leq \alpha$$

for the uppersequent $c : \Gamma_q$ *of the rule* J_q.

Proof. First we show Lemmata 5.7.1 and 5.7.2 simultaneously by induction on the number of sequents between K and J.

Proof of Lemma 5.7.1.

By the definition of the number $m(i)$ we have $i \leq i_{m-1}$. Since the chain $\mathcal{C}_{n_m}$ starting with J_{n_m} passes through the left side of the i_{m-1}-knot K_{m-1}, we have the assertion $\sigma_{n_m+1} \prec_i \sigma_{n_{m-1}+1}$ by IH on Lemma 5.7.2.

$$\dfrac{\dfrac{\vdots\ \mathcal{C}_{n_m}}{\Phi_{m-1}, \neg A_{m-1}} \quad A_{m-1}, \Psi_{m-1}}{\Phi_{m-1}, \Psi_{m-1}}\ (\Sigma_{i_{m-1}})^{\sigma_{n_{m-1}+1}}\, K_{m-1}$$
$$\vdots$$
$$\dfrac{\Gamma_{n_m}}{\Gamma'_{n_m}}\ (c)_{\sigma_{n_m+1}}\, J_{n_m}$$

This shows Lemma 5.7.1. □

Proof of Lemma 5.7.2.

By **(ch:Qpt)** we have

$$pd_i(\rho) = \sigma_{n_{m(i)}+1} \text{ and } J_{n_{m(i)}} = pd_i(J).$$

Claim 5.2 $a \leq n_{m(i)} + 1$, *i.e.,* $J_{n_{m(i)}+1} \subset K$.

Proof of Claim 5.2. If $m(i) = l$, then $a \leq n = n_l + 1$. Assume

$$m(i) < l \neq 0 \,\&\, a > n_{m(i)} + 1.$$

Then the $i_{m(i)}$-knot $(\Sigma_{i_{m(i)}})^{\sigma_{n_{m(i)}+1}} K_{m(i)}$ is above the left uppersequent of K, $K * (0) \subset K_{m(i)}$, and $j \geq i > i_{m(i)}$. Consider the left rope ${}_{K_{m(i)}}\mathcal{R} = J_{n_{m(i)}+1}, \ldots, J_{b-1}$ of the knot $K_{m(i)}$ for the chain $\mathcal{C}$. Then by Lemma 5.6 we have $b = n$. Therefore ${}_{K_{m(i)}}\mathcal{R}$ reaches to the rule K. Thus by **(uplwl)** we have $i \leq j < i_{m(i)}$. This is a contradiction. □

By the Claim 5.2 we have Lemma 5.7.2b.
Case 1 $a = n_{m(i)} + 1$: This means that the i-predecessor $J_{n_{m(i)}}$ of J is the rule J_{a-1}, and $pd_i(\rho) = \sigma_a$.
Case 2 $a < n_{m(i)} + 1$: This means that $J_{n_{m(i)}} \subset K$. Put

$$m_1 = \min\{m \leq m(i) : a < n_m + 1\} \tag{15}$$

Then $J_{n_{m_1}}$ is the uppermost rule J_{n_m} below K. The chain $\mathcal{C}_{n_{m_1}}$ starting with $J_{n_{m_1}}$ passes through the left side of the knot $(\Sigma_{i_{m_1-1}})^{\sigma_{n_{m_1-1}+1}}\, K_{m_1-1}$. If $K \subset K_{m_1-1}$, then $\mathcal{C}_{n_{m_1}}$ passes through the left side of K.

$$\begin{array}{c}
\cfrac{\begin{array}{cc}\vdots\ \mathcal{C}_{n_{m_1}} & \vdots \\ \Phi, \neg A & A, \Psi\end{array}}{\Phi, \Psi}\ (\Sigma_j)^{\sigma_a}\ K \\
\vdots \\
\cfrac{\Gamma_{n_{m_1}}}{\Gamma'_{n_{m_1}}}\ (c)_{\sigma_{n_{m_1}+1}}\ J_{n_{m_1}}
\end{array}$$

And by the minimality of m_1, if $K_{m_1-1} \subset K$, then $J_{a-1} = J_{n_{m_1-1}}$, i.e., $a = n_{m_1-1} + 1$.

$$\begin{array}{c}
\vdots \\
\cfrac{\Gamma_{a-1}}{\Gamma'_{a-1}}\ J_{a-1} = J_{n_{m_1-1}} \\
\cfrac{\begin{array}{cc}\vdots\ \mathcal{C} & \vdots \\ \Phi, \neg A & A, \Psi\end{array}}{\Phi, \Psi}\ (\Sigma_j)^{\sigma_a}\ K \\
\cfrac{\begin{array}{cc}\vdots\ \mathcal{C}_{n_{m_1}} & \vdots\ \mathcal{C} \\ \Phi_{m_1-1}, \neg A_{m_1-1} & A_{m_1-1}, \Psi_{m_1-1}\end{array}}{\Phi_{m_1-1}, \Psi_{m_1-1}}\ (\Sigma_{i_{m_1-1}})^{\sigma_a}\ K_{m_1-1} \\
\vdots\ \mathcal{C}_{n_{m_1}} \\
\cfrac{\Gamma_{n_{m_1}}}{\Gamma'_{n_{m_1}}}\ (c)_{\sigma_{n_{m_1}+1}}\ J_{n_{m_1}}
\end{array}$$

By Lemma 5.7.1 we have $pd_i(\rho) = \sigma_{n_{m(i)}+1} \preceq_i \sigma_{n_{m_1}+1}$. Once again by IH we have $\sigma_{n_{m_1}+1} \preceq_i \sigma_a$. Thus we have shown Lemma 5.7.2a, $\rho \prec_i \sigma_a$. □

Proof of Lemma 5.7.2c. $j < i_p$: This is seen from **(uplwl)** as in the proof of the Claim 5.2 since in this case we have $l \neq 0$.
A proof of Lemma 5.7.2 is completed. □

Proof of Lemma 5.7.3.
The chain $\mathcal{C}_{b-1}$ passes through the left side of K and $\mathcal{C}$ the right side of K. By **(ch:Qpt)** we have

$$pd_i(\sigma_{n+1}) = \sigma_{n_{m(i)}+1} \text{ and } J_{n_{m(i)}} = pd_i(J_n).$$

If K is a K_m for an $m < l$, then the assertion $i \leq j = i_m$ follows from (13) since $b-1 \leq n_{m(i)}$ by $\sigma_{n+1} \prec_i \sigma_b$, and hence $m < m(i)$.

Otherwise let $m \leq m(i)$ denote the number such that $n_m > b-1 > n_{m-1}$, i.e., J_{b-1} is between K_{m-1} and J_{n_m}. Then K is below K_{m-1} and the rule K is the merging rule of $\mathcal{C}_{b-1}$ and the chain $\mathcal{C}_{n_m}$ starting with J_{n_m}, i.e., $\mathcal{C}_{n_m}$ passes through the right side of K.

$$
\begin{array}{c}
\dfrac{
\begin{array}{c}\vdots\ \mathcal{C}_{b-1}\\ \Phi,\neg A\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ \mathcal{C}_{n_m}\\ \Phi_{m-1},\neg A_{m-1}\end{array}\quad \begin{array}{c}\vdots\ \mathcal{C}\\ A_{m-1},\Psi_{m-1}\end{array}}{\Phi_{m-1},\Psi_{m-1}}\,K_{m-1}\\
\vdots\ \mathcal{C}_{n_m},\mathcal{C}\\
A,\Psi
\end{array}
}{\Phi,\Psi}\,(\Sigma_j)\,K \\
\vdots \\
\dfrac{\Gamma_{b-1}}{\Gamma'_{b-1}}\ (c)_{\sigma_b}\ J_{b-1} \\
\vdots \\
\dfrac{\Gamma_{n_m}}{\Gamma'_{n_m}}\ (c)_{\sigma_{n_m+1}}\ J_{n_m}
\end{array}
$$

By IH it suffices to show that $\sigma_{n_m+1} \prec_i \sigma_b$ and this follows from

$$\sigma_{n+1} \prec_i \sigma_{n_m+1} \tag{16}$$

since the set $\{\tau : \sigma_{n+1} \prec_i \tau\}$ is linearly ordered by $\prec_i$, Proposition 4.1.1. Now (16) follows from (13) and Lemma 5.7.2a, i.e,

$$\sigma_{n+1} \prec_i pd_i(\sigma_{n+1}) = \sigma_{n_{m(i)}+1} \prec_i \cdots \prec_i \sigma_{n_{m-1}+1} \prec_i \sigma_{n_m+1}.$$

This shows Lemma 5.7.3. $\square$

Proof of Lemma 5.7.4 by induction on the number of sequents between K and J.

By the Claim 5.2 we have $a \leq n_{m(i)} + 1$.

Case 1 $a = n_{m(i)} + 1$: This means that the i-predecessor $J_{n_{m(i)}}$ of J is the rule J_{a-1} and $pd_i(\rho) = \sigma_a$. By Lemma 5.7.2c we have $i_p > j \geq i$ for any $p < m(i)$. On the other side by **(ch:Qpt)**

$$i \in In(\mathcal{C}) = In(\rho) \Leftrightarrow \exists p \in [0, m(i))(i_p = i) \tag{17}$$

Hence $i \notin In(\rho)$. Thus $in_i(\sigma_a) = in_i(pd_i(\rho)) = in_i(\rho)$.

Case 2 $a < n_{m(i)} + 1$: This means that $J_{n_{m(i)}}$ is below K. Let m_1 denote the number (15) defined in the proof of Lemma 5.7.2.

Claim 5.3 *For each $m \in (m_1, m(i)]$ the i-origin of J_{n_m} is not below K_{m-1}, $i < i_{m-1}$, $i \notin In(\rho)$, $\sigma_{n_m+1} \prec_i \sigma_{n_{m-1}+1}$ and $\forall b \in (n_{m-1}+1, n_m+1]\{\sigma_{n_m+1} \preceq_i \sigma_b \prec_i \sigma_{n_{m-1}+1} \to i \notin In(\sigma_b)\}$ and*

$$
\begin{aligned}
&\forall b \in (n_{m-1}+1, n_m+1]\{\sigma_{n_m+1} \preceq_i \sigma_b \prec_i \sigma_{n_{m-1}+1} \to i \notin In(\sigma_b)\} \\
&\forall b \in (n_{m-1}+1, n_m+1]\{\sigma_{n_m+1} \preceq_i \sigma_b \prec_i \sigma_{n_{m-1}+1} \to \\
&in_i(J_{n_m}) = in_i(J_{b-1}) = in_i(J_{n_{m-1}})\ ,\ i.e., \\
&in_i(\sigma_{n_m+1}) = in_i(\sigma_b) = in_i(\sigma_{n_{m-1}+1})\}
\end{aligned} \tag{18}
$$

Proof of the Claim 5.3. First we show $i < i_{m-1}$. By Lemma 5.7.1 we have $i \leq i_{m-1}$. Assume $i = i_{m-1}$ for some $m \in (m_1, m(i)]$. Pick the minimal such m_2. Then by **(ch:Qpt)**, (17) we have $i \in In(\rho)$ and hence $rg_i(\rho)\downarrow$. By Lemma 5.7.2c we have

$$p < m_1 \Rightarrow i_p > j \geq i \tag{19}$$

Here $p < m_1$ means that K_p is above K. Thus by **(ch:Qpt)**

$$m(i+1) = \max\{m : 0 \leq m \leq l \,\&\, \forall p \in [0, m)(i+1 \leq i_p)\} = m_2 - 1 \geq m_1,$$

i.e.,

$$J_{n_{m_2-1}} = pd_{i+1}(J) \,\&\, \sigma_{n_{m_2-1}+1} = pd_{i+1}(\rho) \,\&\, n_{m_2-1} \geq n_{m_1} \geq a.$$

On the other hand by **(ch:Qpt)** we have for the i-origin J_q of $\mathcal{C}$, i.e., $\sigma_q = rg_i(\rho)$,

$$n_{m_2-1} = n_{m(i+1)} < q \leq n_{m(i)} + 1.$$

Thus J_q is below $J_{n_{m_2-1}}$ and hence by $a \leq n_{m_2-1}$ the i-origin J_q is below K. This contradicts our hypothesis.

$$
\begin{array}{c}
\dfrac{\Phi, \neg A \quad A, \Psi}{\Phi, \Psi}\ K \\
\vdots \\
\dfrac{\Gamma_{n_{m_2-1}}}{\Gamma'_{n_{m_2-1}}}\ (c)_{pd_{i+1}(\rho)}\ J_{n_{m_2-1}} \\
\vdots \\
\dfrac{\Phi_{m_2-1}, \neg A_{m_2-1} \quad A_{m_2-1}, \Psi_{m_2-1}}{\Phi_{m_2-1}, \Psi_{m_2-1}}\ (\Sigma_i)\ K_{m_2-1} \\
\vdots \\
\dfrac{\Gamma_q}{\Gamma'_q}\ (c)^{rg_i(\rho)}\ J_q \\
\vdots \\
\dfrac{\Gamma_{n_{m(i)}}}{\Gamma'_{n_{m(i)}}}\ (c)_{pd_i(\rho)}\ J_{n_{m(i)}}
\end{array}
$$

Thus we have shown $i < i_{m-1}$ for any $m \in (m_1, m(i)]$. From this, (19) and (17) we see $i \notin In(\rho)$ and hence

$$in_i(\rho) = in_i(pd_i(\rho)) = in_i(\sigma_{n_{m(i)}+1}).$$

In particular, if $rg_i(\rho) \downarrow$, then $\sigma_q = rg_i(\rho) = rg_i(\sigma_{n_{m(i)}+1})$, i.e., the i-origin of $J_{n_{m(i)}}$ equals to the i-origin J_q of J. Therefore the i-origin of $J_{n_{m(i)}}$ is not below K and a fortiori not below the $i_{m(i)-1}$-knot $K_{m(i)-1}$. Also the chain starting with $J_{n_{m(i)}}$ passes through the left side of $K_{m(i)-1}$ and $i \leq i_{m(i)-1}$. Thus by IH we have (18) for $m = m(i)$. We see similarly that for each $m \in (m_1, m(i)]$ the i-origin of J_{n_m} is not below K_{m-1} and (18).

Thus we have shown the Claim 5.3. □

From Claim 5.3 we see

$$\forall b \in (n_{m_1}+1, n+1]\{\rho \preceq_i \sigma_b \prec_i \sigma_{n_{m_1}+1} \to i \notin In(\sigma_b)\}$$

and hence

$$\forall b \in (n_{m_1}+1, n+1]\{\rho \preceq_i \sigma_b \prec_i \sigma_{n_{m_1}+1} \to$$
$$in_i(J) = in_i(J_{b-1}) = in_i(J_{n_{m_1}}), \text{ i.e., } in_i(\rho) = in_i(\sigma_b) = in_i(\sigma_{n_{m_1}+1})\}.$$

Further

$$\forall m \in (m_1, m(i)]\{\rho \prec_i \sigma_{n_m+1} \prec_i \sigma_{n_{m-1}+1} \preceq_i \sigma_{n_{m_1}+1}\}.$$

Once more by IH we have, cf. Figures in the proof of Lemma 5.7.2, **Case 2.**, $\sigma_{n_{m_1}+1} \preceq_i \sigma_a$ and

$$\forall b \in (a, n_{m_1}+1]\{\sigma_{n_{m_1}+1} \preceq_i \sigma_b \prec_i \sigma_a \to i \notin In(\sigma_b)\}$$

and hence

$$\forall b \in (a, n_{m_1}+1]\{\sigma_{n_{m_1}+1} \preceq_i \sigma_b \prec_i \sigma_a \to$$
$$in_i(J_{n_{m_1}}) = in_i(J_{b-1}) = in_i(J_{a-1}), \text{ i.e., } in_i(\sigma_{n_{m_1}+1}) = in_i(\sigma_b) = in_i(\sigma_a)\}$$

Thus we have shown Lemma 5.7.4. □

Proof of Lemma 5.7.5 by induction on the number of sequents between K and J_{b-1}.

Let $\mathcal{C}_b = I_0, \ldots, I_{b-1}$ denote the chain starting with $J_{b-1} = I_{b-1}$. Each rule I_p is again a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$. Chains $\mathcal{C}_b$ and $\mathcal{C}$ intersect in a way described as **Type1 (segment)** or **Type3 (merge)** in **(ch:link)**. If the chain $\mathcal{C}_b$ passes through the left side of K, then the i-origin I_q of $\mathcal{C}_b$ is above K if it exists, and hence the assertion follows from Lemma 5.7.4.

Otherwise there exists a merging rule $(\Sigma_l)^{\sigma_c}$ I below K such that the chain

$\mathcal{C}_b$ passes through the left side of I and $\mathcal{C}$ the right side of I.

$$
\begin{array}{c}
\dfrac{\begin{array}{cc}\vdots\ \mathcal{C} & \vdots \\ \Phi,\neg A & A,\Psi\end{array}}{\Phi,\Psi}\ K \\
\vdots \\
\dfrac{\Gamma_{c-1}}{\Gamma'_{c-1}}\ (c)_{\sigma_c}\ J_{c-1} \\
\dfrac{\begin{array}{cc}\vdots\ \mathcal{C}_b & \vdots\ \mathcal{C} \\ \Pi,\neg B & B,\Delta\end{array}}{\Pi,\Delta}\ (\Sigma_l)^{\sigma_c}\ I \\
\vdots \\
\dfrac{\Gamma_{b-1}}{\Gamma'_{b-1}}\ (c)_{\sigma}\ J_{b-1}
\end{array}
$$

Then by Lemma 5.7.3 we have $i \leq l$. The i-origin I_q of $\mathcal{C}_b$ is not below I. Therefore by Lemma 5.7.4 we have

$$\forall d \in (c,b]\{\sigma \preceq_i \sigma_d \prec_i \sigma_c \to i \notin In(\sigma_d)\}$$

and

$$\sigma \prec_i \sigma_c.$$

Hence

$$\forall d \in (c,b]\{\sigma \preceq_i \sigma_d \prec_i \sigma_c \to in_i(\sigma_d) = in_i(\sigma_c)\}.$$

In particular

$$in_i(\sigma_c) = in_i(\sigma)\,\&\,\sigma \prec_i \sigma_c \tag{20}$$

Now consider the member J_{c-1} of $\mathcal{C}$. J_{c-1} is again below K, $\sigma_{n+1} \preceq_i \sigma_c$ and $rg_i(\sigma_c) \simeq rg_i(\sigma)$ by (20). Thus by IH we have

$$\forall d \in (a,c]\{\sigma_c \preceq_i \sigma_d \prec_i \sigma_a \to i \notin In(\sigma_d)\}$$

and

$$\sigma_c \prec_i \sigma_a.$$

Therefore

$$\forall d \in (a,c]\{\sigma_c \preceq_i \sigma_d \prec_i \sigma_a \to in_i(\sigma_d) = in_i(\sigma_a)\}.$$

This shows Lemma 5.7.5. □

Proof of Lemma 5.7.6.

Pick a p_0 so that $n \geq p_0 \geq q_0$, $\rho \preceq_i \sigma_{p_0+1}$ and $rg_i(\sigma_{p_0+1}) = \sigma_{q_0} = \kappa$.

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\vdots \\
\dfrac{\Gamma_{a-1}}{\Gamma'_{a-1}}\ (c)_{\sigma_a}\ J_{a-1} \\
\vdots\ \mathcal{C} \\
\Phi, \neg A
\end{array}
&
\begin{array}{c}
\vdots \\
A, \Psi
\end{array}
\end{array} \\
\hline
\Phi, \Psi
\end{array}
\ (\Sigma_j)^{\sigma_a}\ K
$$

$$
\begin{array}{c}
\vdots \\
\dfrac{\Gamma_{q_0}}{\Gamma'_{q_0}}\ (c)^{\kappa}\ J_{q_0}\ (rg_i(\sigma_{p_0+1}) = \kappa) \\
\vdots \\
\dfrac{\Gamma_{p_0}}{\Gamma'_{p_0}}\ (c)_{\sigma_{p_0+1}}\ J_{p_0} \\
\vdots \\
\dfrac{\Gamma_{n}}{\Gamma'_{n}}\ (c)_{\rho}\ J_{n}
\end{array}
$$

Lemma 5.7.6a. First note that $\rho \preceq_i \sigma_{p_0+1} \prec_i rg_i(\sigma_{p_0+1}) = \kappa$ by Proposition 4.1.2 (or by the proviso **(ch:Qpt)**) and hence $\rho \prec_i \kappa$. Thus the assertion follows from Lemma 5.7.5 and the minimality of q_0.

Lemma 5.7.6b. Suppose $rg_i(\sigma_t) \not\preceq_i \kappa$ for a t with $\rho \preceq_i \sigma_t \prec_i \kappa$. Put $\sigma_b = rg_i(\sigma_t)$. Then by Propositions 4.1.1 and 4.1.2 we have $\kappa = \sigma_{q_0} \prec_i \sigma_b$ and $b < q_0 < t \,\&\, q_0 \geq a$. Hence by the minimality of q_0 we have $b < a$.

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\vdots \\
\dfrac{\Gamma_{b}}{\Gamma'_{b}}\ (c)^{\sigma_b}\ J_{b}\ (\sigma_b = rg_i(\sigma_t)) \\
\vdots\ \mathcal{C} \\
\Phi, \neg A
\end{array}
&
\begin{array}{c}
\vdots \\
A, \Psi
\end{array}
\end{array} \\
\hline
\Phi, \Psi
\end{array}
\ (\Sigma_j)^{\sigma_a}\ K
$$

$$
\begin{array}{c}
\vdots \\
\dfrac{\Gamma_{q_0}}{\Gamma'_{q_0}}\ (c)^{\kappa}\ J_{q_0} \\
\vdots \\
\dfrac{\Gamma_{t-1}}{\Gamma'_{t-1}}\ (c)_{\sigma_t}\ J_{t-1}
\end{array}
$$

Thus by Lemma 5.7.5 we have

$$in_i(\sigma_a) = in_i(\sigma_t).$$

From this and Lemma 5.7.6a we have

$$in_i(\sigma_a) = in_i(\sigma_t) = in_i(\kappa) \,\&\, \sigma_t \prec_i \kappa \preceq_i \sigma_a \tag{21}$$

Case 1 $t \leq p_0$: Then $\sigma_{p_0+1} \prec_i \sigma_t \prec_i \kappa = rg_i(\sigma_{p_0+1})$ by Proposition 4.1.1. By Proposition 4.1.4 we would have $\sigma_b = rg_i(\sigma_t) \preceq_i \kappa$. Thus this is not the case.

Alternatively we can handle this case without appealing Proposition 4.1.4 as follows. Let p_0 denote the minimal p_0 such that

$$n \geq p_0 \geq q_0 \,\&\, \rho \preceq_i \sigma_{p_0+1} \,\&\, rg_i(\sigma_{p_0+1}) = \sigma_{q_0} = \kappa.$$

Then by **(ch:Qpt)** we have $\kappa = rg_i(\sigma_{p_0+1}) = pd_i(\sigma_{p_0+1})$ and hence this is not the case, i.e., $p_0 < t$.

Case 2 $p_0 < t$: Then $\sigma_t \preceq_i \sigma_{p_0+1} \prec_i \kappa$. By (21) and Proposition 4.1.3, or by Lemma 5.7.5 we would have $in_i(\sigma_{p_0+1}) = in_i(\kappa)$. In particular $\kappa = rg_i(\sigma_{p_0+1}) = rg_i(\kappa)$ but $rg_i(\kappa)$ is a proper subdiagram of κ. This is a contradiction.

This shows Lemma 5.7.6b. □

Proof of Lemma 5.7.7 by induction on the number of sequents between K and J_{b-1}.

Case 1 J_q is the i-origin of J_{b-1}, i.e., J_q is a member of the chain starting with $(c)_\sigma\, J_{b-1}$: By the proviso **(st:bound)** we can assume $i \notin In(\sigma)$. Then $in_i(\sigma) = in_i(pd_i(\sigma)) = in_i(J_p)$ with $J_p = pd_i(J_{b-1}) \,\&\, a \leq b-1 < p$ by Lemma 5.7.2. In particular $rg_i(\sigma) = rg_i(pd_i(\sigma))$. IH and $st_i(pd_i(\sigma)) = st_i(\sigma)$ yields the lemma.

Case 2 Otherwise: First note that $\sigma_{n+1} \neq \sigma$ and $\sigma_{n+1} \prec_i \sigma$. By **(ch:Qpt)** we have

$$pd_i(\sigma_{n+1}) = \sigma_{n_{m(i)}+1} \text{ and } J_{n_{m(i)}} = pd_i(J_n).$$

Also $\sigma_{n_{m(i)}+1} \preceq_i \sigma$ and hence $\sigma_{n_{m(i)}+1} \leq \sigma$. Let m_1 denote the number such that

$$m_1 = \min\{m : \sigma_{n_m+1} \leq \sigma\} \leq m(i).$$

Then the rule $(c)_\sigma\, J_{b-1}$ is a member of the chain $\mathcal{C}_{n_{m_1}}$ starting with $J_{n_{m_1}}$ and J_{b-1} is below $(\Sigma_{i_{m_1-1}})\, K_{m_1-1}$. Also the chain $\mathcal{C}_{n_{m_1}}$ passes thorugh the left side of the knot K_{m_1-1}. By $m_1 \leq m(i)$ and Lemma 5.7.1 we have $\sigma_{n_{m(i)}+1} \preceq_i \sigma_{n_{m_1}}$ and hence

$$i \leq i_{m_1-1} \,\&\, \sigma_{n_{m_1}} \preceq_i \sigma. \tag{22}$$

Case 2.1 J_q is below K_{m_1-1}, i.e., $\sigma_q \leq \sigma_{n_{m_1-1}+1}$, i.e., $n_{m_1-1} < q$: By IH and (22) we get the assertion.

Case 2.2 Otherwise: By Lemma 5.7.5 and (22) we have

$$in_i(\sigma) = in_i(\sigma_{n_{m_1}+1}) \,\&\, \sigma \prec_i \sigma_{n_{m_1}+1}.$$

Hence $st_i(\sigma) = st_i(\sigma_{n_{m_1}+1}) \,\&\, rg_i(\sigma) = rg_i(\sigma_{n_{m_1}+1})$. IH and $st_i(\sigma_{n_{m_1}+1}) = st_i(\sigma)$ yield the lemma. □

Lemma 5.8 *Let $\mathcal{R} = J_0, \ldots, J_{n-1}$ denote the rope starting with a top $(c)^\pi\, J_0$. Each J_p is a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$. Let*

$$0 \le n_0 < n_1 < \cdots < n_l = n - 1\, (l \ge 0) \tag{11}$$

be the knotting numbers of the rope $\mathcal{R}$, and K_m an i_m-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}$ of J_{n_m} : and J_{n_m+1} for $m < l$. For $2 \le i < N$ let $m(i)$ denote the number

$$m(i) = \max\{m : 0 \le m \le l \,\&\, \forall p \in [0, m)(i \le i_p)\} \tag{13}$$

Note that $i_m \le N - 2$ by **(ch:left)**. *Also put* (cf. **(ch:Qpt)**)

1.
$$pd_i = \sigma_{n_{m(i)}+1}.$$

2.
$$i \in In \Leftrightarrow \exists p \in [0, m(i))(i_p = i).$$

3. *For $i \in In$ $(i \ne N-1)$,*

   Case 1 *The case when there exists a q such that*

   $$\exists p[n_{m(i)} \ge p \ge q > n_{m(i+1)} \,\&\, pd_i \preceq_i \sigma_{p+1} \,\&\, \sigma_q = rg_i(\sigma_{p+1})] \tag{23}$$

   Then
   $$rg_i = \sigma_q$$
   where q denotes the minimal q satisfying (23).

   Case 2 *Otherwise.*
   $$rg_i = pd_i = \sigma_{n_{m(i)}+1}.$$

1. *For each $i \in In$ we have*
   (a) $in_i(rg_i) = in_i(pd_{i+1}) \,\&\, pd_i \preceq_i rg_i \preceq_i pd_{i+1} \,\&\, pd_i \ne pd_{i+1}$.
   (b) $\forall t[rg_i(pd_i) \preceq_i \sigma_t \prec_i rg_i \Rightarrow rg_i(\sigma_t) \preceq_i rg_i]$.
   (c) *Either* $rg_i = pd_i$ *or* $rg_i(pd_i) \preceq_i rg_i$.

2. *Assume $i \in In \,\&\, \sigma_q := rg_i \ne pd_i$, i.e.,* **Case 1** *occurs. Then*
   $$B_{\sigma_q}(c; P) \le \alpha$$
   for the uppersequent $c : \Gamma_q$ of the rule J_q, $st_i(\sigma_{p+1}) = d_{\sigma_q^+}\alpha$ and p denotes a number such that
   $$n_{m(i)} \ge p \ge q > n_{m(i+1)} \,\&\, pd_i = \sigma_{n_{m(i)}+1} \preceq_i \sigma_{p+1} \,\&\, \sigma_q = rg_i(\sigma_{p+1}).$$

Proof.

Lemma 5.8.1.

Let $i \in In$, and put $\sigma_{q_0} = rg_i \,\&\, \sigma_{p_0} = pd_i \,\&\, \sigma_r = pd_{i+1}$. By the definition we have $p_0 = n_{m(i)}+1 \,\&\, r = n_{m(i+1)}+1$, $m(i) > m(i+1) \,\&\, i_{m(i+1)} = i$, $p_0 \leq q_0 \leq r$ and $\sigma_{p_0} \preceq_i \sigma_{q_0}$. Also

$$\forall p \in [m(i+1), m(i))(i \leq i_p).$$

From this and Lemma 5.7.1 we see

$$\forall p \in [m(i+1), m(i))(\sigma_{n_{p+1}+1} \prec_i \sigma_{n_p+1}) \tag{24}$$

On the other hand we have by the definition of $rg_i = \sigma_{q_0}$

$$\neg\exists q < q_0 \exists p[p_0 - 1 \leq p \leq q > r - 1 \,\&\, pd_i \preceq_i \sigma_{p+1} \,\&\, \sigma_q = rg_i(\sigma_p)] \tag{25}$$

Case 2. Then $pd_i = rg_i$, i.e., $p_0 = q_0$, and Lemma 5.8.1b vacuously holds. Lemma 5.8.1a, $in_i(\sigma_{q_0}) = in_i(\sigma_r) \,\&\, \sigma_{q_0} \preceq_i \sigma_r$, follows from (24) and Lemma 5.7.4 with (25).

Case 1. Let m denote the number such that

$$m(i) \geq m > m(i+1) \,\&\, n_m \geq q_0 > n_{m-1} \tag{26}$$

i.e., the rule $(c)^{\sigma_{q_0}} J_{q_0}$ is a member of the chain $\mathcal{C}_{n_m}$ starting with J_{n_m}.

Claim 5.4 *Let p_1 denote the minimal p_1 such that $\sigma_{p_0} \preceq_i \sigma_{p_1+1}$ and $\sigma_{q_0} = rg_i(\sigma_{p_1+1})$. Then $p_1 \leq n_m \,\&\, \sigma_{n_m+1} \preceq_i \sigma_{p_1+1}$.*

$$
\begin{array}{c}
\vdots\, \mathcal{C}_{n_m} \qquad\qquad \vdots \\
\dfrac{\Phi_{m-1}, \neg A_{m-1} \quad A_{m-1}, \Psi_{m-1}}{\Phi_{m-1}, \Psi_{m-1}}\, K_{m-1} \\
\vdots \\
\dfrac{\Gamma_{q_0}}{\Gamma'_{q_0}}\, (c)^{rg_i(\sigma_{p_1+1})}\, J_{q_0} \\
\vdots \\
\dfrac{\Gamma_{p_1}}{\Gamma'_{p_1}}\, (c)_{\sigma_{p_1+1}}\, J_{p_1} \\
\vdots \\
\dfrac{\Gamma_{n_m}}{\Gamma'_{n_m}}\, (c)_{\sigma_{n_m+1}}\, J_{n_m}
\end{array}
$$

Proof of Claim 5.4. Let m_1 denote the number such that

$$m(i) \geq m_1 > m(i+1) \,\&\, n_{m_1} \geq p_1 > n_{m_1-1}.$$

Then by (24), $pd_i \preceq_i \sigma_{n_{m_1}+1}$ and $pd_i \preceq_i \sigma_{p_1+1}$ we have $\sigma_{n_{m_1}+1} \preceq_i \sigma_{p_1+1}$. It remains to show $m = m_1$. Assume $m < m_1$. Then by Lemma 5.7.5 and $q_0 < n_{m_1-1}+1$ we would have $in_i(\sigma_{n_{m_1-1}+1}) = in_i(\sigma_{p_1+1})$ and hence $rg_i(\sigma_{n_{m_1-1}+1}) = rg_i(\sigma_{p_1+1}) = \sigma_{q_0}$. This contradicts the minimality of p_1 by (24).

$$
\begin{array}{c}
\vdots\, \mathcal{C}_{n_m} \qquad\qquad\quad \vdots \\
\dfrac{\Phi_{m-1}, \neg A_{m-1} \quad A_{m-1}, \Psi_{m-1}}{\Phi_{m-1}, \Psi_{m-1}}\ K_{m-1} \\
\vdots \\
\dfrac{\Gamma_{q_0}}{\Gamma'_{q_0}}\ (c)^{rg_i(\sigma_{p_1+1})}\ J_{q_0} \\
\vdots \\
\dfrac{\Gamma_{n_m}}{\Gamma'_{n_m}}\ J_{n_m} \\
\vdots\, \mathcal{C}_{n_{m_1}} \qquad\qquad\qquad \vdots \\
\dfrac{\Phi_{m_1-1}, \neg A_{m_1-1} \quad A_{m_1-1}, \Psi_{m_1-1}}{\Phi_{m_1-1}, \Psi_{m_1-1}}\ (\Sigma_{i_{m_1-1}})^{\sigma_{n_{m_1-1}}}\ K_{m_1-1} \\
\vdots \\
\dfrac{\Gamma_{p_1}}{\Gamma'_{p_1}}\ (c)_{\sigma_{p_1+1}}\ J_{p_1} \\
\vdots \\
\dfrac{\Gamma_{n_{m_1}}}{\Gamma'_{n_{m_1}}}\ J_{n_{m_1}}
\end{array}
$$

This shows Claim 5.4. □

By the minimality of q_0 and the Claim 5.4 q_0 is the minimal q such that

$$\exists p[n_m \geq p \geq q \geq n_{m-1}+1 \,\&\, \sigma_{n_m+1} \preceq_i \sigma_{p+1} \,\&\, \sigma_q = rg_i(\sigma_{p+1})].$$

Hence by Lemma 5.7.6 we have

$$in_i(\sigma_{n_{m-1}+1}) = in_i(\sigma_{q_0}) \,\&\, \sigma_{q_0} \preceq_i \sigma_{n_{m-1}+1} \tag{27}$$

and

$$\forall t[\sigma_{n_m+1} \preceq_i \sigma_t \prec_i \sigma_{q_0} \Rightarrow rg_i(\sigma_t) \preceq_i \sigma_{q_0}] \tag{28}$$

Lemma 5.8.1a. By (27) it suffices to show that

$$in_i(\sigma_{n_{m-1}+1}) = in_i(pd_{i+1}) \,\&\, \sigma_{n_{m-1}+1} \preceq_i pd_{i+1}.$$

This follows from (24) and Lemma 5.7.4 with (25).

Lemma 5.8.1b and 5.8.1c. In view of (28) it suffices to show the

Claim 5.5 $\sigma_{p_0} \preceq_i \sigma_t \prec_i \sigma_{n_m+1} \Rightarrow rg_i(\sigma_t) \preceq_i \sigma_{q_0}$.

Proof of Claim 5.5 by induction on t with $p_0 > t > n_m + 1$.

Let $m_2 \geq m$ denote the number such that $n_{m_2+1} \geq t > n_{m_2}$. Then the chain $\mathcal{C}_{n_{m_2+1}}$ starting with $J_{n_{m_2+1}}$ passes through the left side of the rule $(\Sigma_{i_{m_2}})^{\sigma_{n_{m_2}+1}} K_{m_2}$.

$$
\dfrac{\dfrac{\dfrac{\begin{array}{cc} \vdots\, \mathcal{C}_{n_{m_2+1}} & \vdots \\ \Phi_{m_2}, \neg A_{m_2} & A_{m_2}, \Psi_{m_2} \end{array}}{\Phi_{m_2}, \Psi_{m_2}} \; (\Sigma_{i_{m_2}})^{\sigma_{n_{m_2}+1}} K_{m_2}}{\begin{array}{c} \vdots \\ \Gamma_{t-1} \end{array}}}{\Gamma'_{t-1}} \; (c)_{\sigma_t} \, J_{t-1}
$$

$$
\begin{array}{c} \vdots \\ \dfrac{\Gamma_{n_{m_2+1}}}{\Gamma'_{n_{m_2+1}}} \; (c)_{\sigma_{n_{m_2+1}+1}} \, J_{n_{m_2+1}} \end{array}
$$

We have

$$
\sigma_{n_{m_2+1}+1} \preceq_i \sigma_t \prec_i \sigma_{n_{m_2}+1} \tag{29}
$$

by (24). Put $\sigma_b = rg_i(\sigma_t)$. It suffices to show $b \geq q_0$.

First consider the case when $b \leq n_{m_2}$. Then by (29) and Lemma 5.7.5 we have $in_i(\sigma_t) = in_i(\sigma_{n_{m_2}+1})$ and $rg_i(\sigma_t) = rg_i(\sigma_{n_{m_2}+1})$. Thus IH when $m_2 > m$ and (28) when $m_2 = m$ yield $b \geq q_0$.

Next suppose $b > n_{m_2}$. Let $q_1 \leq b$ denote the minimal $q \leq b$ such that

$$
\exists p[n_{m_2+1} \geq p \geq q \geq n_{m_2} + 1 \,\&\, \sigma_{n_{m_2+1}} \preceq_i \sigma_{p+1} \,\&\, \sigma_q = rg_i(\sigma_{p+1})].
$$

The pair $(p, q) = (t - 1, b)$ enjoys this condition.

Then by Lemma 5.7.6 we have $\sigma_{q_1} \preceq_i \sigma_{n_{m_2}+1}$. Thus $\sigma_b \preceq_i \sigma_{q_1} \preceq_i \sigma_{n_{m_2}+1} \preceq_i \sigma_{n_m+1} \preceq_i \sigma_{q_0}$. This shows Claim 5.5. □

Lemma 5.8.2.

First observe that as in (22) in the proof of Lemma 5.7.7,

$$
\forall m \leq m(i)[\sigma_{n_{m(i)}+1} \preceq_i \sigma_{n_m+1}] \tag{30}
$$

Put

$$
\begin{aligned}
m_1 &= \min\{m : p \leq n_m\} \\
m_2 &= \min\{m : q \leq n_m\}.
\end{aligned}
$$

Then the rule J_p $[J_q]$ is a member of the chain $\mathcal{C}_{n_{m_1}}$ $[\mathcal{C}_{n_{m_2}}]$ starting with $J_{n_{m_1}}$ [starting with $J_{n_{m_2}}$], resp. and $m(i+1) < m_2 \leq m_1 \leq m(i)$.

Claim 5.6 *(Cf. Claim 5.4.) There exists a p_0 such that*

$$
in_i(\sigma_{p+1}) = in_i(\sigma_{p_0+1}) \,\&\, \sigma_{n_{m_2}+1} \preceq_i \sigma_{p_0+1} \,\&\, n_{m_2} \geq p_0 > n_{m_2-1},
$$

i.e., the rule J_{p_0} is a member of the chain $\mathcal{C}_{n_{m_2}}$ starting with $J_{n_{m_2}}$.

Proof of Claim 5.6.

1. The case $m_1 = m_2$: By (30) and $\sigma_{n_{m(i)}+1} \preceq_i \sigma_{p+1}$ we have

$$\sigma_{n_{m_1}+1} \preceq_i \sigma_{p+1}. \tag{31}$$

   Set $p_0 = p$.

2. The case $m_2 < m_1$: By (31) and Lemma 5.7.5 we have

$$in_i(\sigma_{p+1}) = in_i(\sigma_{n_{m_1-1}+1}) = \cdots = in_i(\sigma_{n_{m_2}+1})$$

   and

$$\sigma_{p+1} \prec_i \sigma_{n_{m_1-1}+1} \prec_i \cdots \prec_i \sigma_{n_{m_2}+1}.$$

   Set $p_0 = n_{m_2}$.

This shows the Claim 5.6. □

By the Claim 5.6 and Lemma 5.7.7 we conclude $rg_i(\sigma_{p+1}) = rg_i(\sigma_{p_0+1})$ and

$$B_{\sigma_q}(c;P) \le b(st_i(\sigma_{p_0+1})) = b(st_i(\sigma_{p+1})) = \alpha.$$

□

Main Lemma 5.1 *If P is a proof, then the endsequent of P is true.*

In the next section we prove the Main Lemma 5.1 by a transfinite induction on $o(P) \in Od(\Pi_N) \mid \Omega$.

Assuming the Main Lemma 5.1 we see Theorem 1.1 as in [6], i.e., attach $(h)^\pi$, $(c\Pi_2)^\Omega$ and $(h)^\Omega$ as last rules to a proof P_0 of A^Ω in T_N.

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\vdots\ P_0\\ A^\Omega\end{array}}{A^\Omega}\ (h)^\pi}{A^\alpha}\ (c\Pi_2)^\Omega_\alpha}{A^\alpha}\ (h)^\Omega \qquad P$$

6 Proof of Main Lemma

Throughout this section P denotes a proof with a chain analysis in T_{Nc} and $r : \Gamma_{rdx}$ the redex of P.

M1. The case when $r : \Gamma_{rdx}$ is a lowersequent of an explicit basic rule J.
M2. The case when $r : \Gamma_{rdx}$ is a lowersequent of an $(ind)\, J$.
M3. The case when the redex $r : \Gamma_{rdx}$ is an axiom.
These are treated as in [5], [6].

By virtue of **M1-3** we can assume that the redex $r : \Gamma_{rdx}$ of P is a lowersequent of a rule $J = r * (0)$ such that J is one of the rules (Π_2^Ω-rfl), (Π_N-rfl) or an implicit basic rule.

M4. J is a (Π_2^Ω-rfl). As in [5] introduce a $(c)^\Omega_{d_\Omega \alpha}$ and a (cut).

M5. J is a (Π_N-rfl).
M5.1. There is no rule $(c)^\pi$ below J.

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\\ \neg\exists z(t < z \wedge A^z), \Gamma\end{array}}{r : \Gamma}\, J \\
\vdots \\
\dfrac{a : \Phi}{a_0 : \Lambda}\,(h)^\pi
\end{array}
\qquad\qquad P
$$

where $a : \Phi$ denotes the uppermost sequent below J such that $h(a; P) = \pi$. The sequent $a_0 : \Lambda$ is the lowersequent of the lowermost $(h)^\pi$.

Let P' be the following:

$$
\dfrac{
\begin{array}{c}
\vdots \\
\dfrac{\Gamma}{\Gamma, A}\,(w) \\
\vdots \\
\dfrac{a_1 : \Phi, A}{\dfrac{\Phi, A^\sigma}{\Lambda, A^\sigma}\,(h)^\pi}\,(c\Pi_N)^\pi_\sigma\, J_0
\end{array}
\qquad
\begin{array}{c}
\vdots\; z := \sigma \\
\dfrac{\neg A^\sigma, \Gamma}{\neg A^\sigma, \Gamma}\,(w) \\
\vdots \\
\dfrac{\neg A^\sigma, \Phi}{\dfrac{\neg A^\sigma, \Phi}{\neg A^\sigma, \Lambda}\,(h)^\pi}\,(w)
\end{array}
}{a_0 : \Lambda}\,(\Sigma_{N-1})^\sigma\, J_0'
\qquad\qquad P'
$$

where the o.d. σ in the new $(c\Pi_N)^\pi_\sigma\, J_0$ is defined to be

$$\sigma = d^q_\pi \alpha \text{ with } q = \nu\pi\pi N - 1, \nu = o(a_1; P') \text{ and } \alpha = \pi \cdot o(a_1; P') + \mathcal{K}_\pi(a; P)$$

Namely $In(\sigma) = \{N - 1\}$, $st_{N-1}(\sigma) = \nu$ and $pd_{N-1}(\sigma) = rg_{N-1}(\sigma) = \pi$.

Then as in [6] we see that $\Phi \subseteq \Delta^\sigma$, $\alpha < Bk_\pi(a; P)\,\&\,\sigma < o(a_0; P)$, $\sigma \in Od(\Pi_N)$ and $o(a_0; P') < o(a_0; P)$. Hence $o(P') < o(P)$. Moreover in P, no chain passes through $a_0 : \Lambda$, and the new $(\Sigma_N)^\sigma\, J_0'$ does not split any chain.

M5.2. There exists a rule $(c)^\pi\, J_0$ below J.
Let $\mathcal{R} = J_0, \ldots, J_{n-1}$ denote the rope starting with J_0. The rope $\mathcal{R}$ need not to be a chain as contrasted with [6]. Each rule J_p is a $(c)^{\sigma_p}_{\sigma_{p+1}}$. Put $\sigma = \sigma_n$.

$$
\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma, A & \neg\exists z(t < z \wedge A^z), \Gamma\end{array}}{r : \Gamma}\ J
$$

$$
\vdots
$$

$$
\dfrac{\Gamma_0}{a_0\Gamma_0'}\ (c)^{\pi}_{\sigma_1}\ J_0
$$

$$
\vdots
$$

$$
\dfrac{a_i : \Gamma_i}{\Gamma_i'}\ (c)^{\sigma_i}_{\sigma_{i+1}}\ J_i
$$

$$
\vdots
$$

$$
\dfrac{a_{n-1} : \Gamma_{n-1}}{\Gamma_{n-1}'}\ (c)^{\sigma_{n-1}}_{\sigma}\ J_{n-1}
$$

$$
\dfrac{\vdots}{a_n : \Gamma_n}\ (\Sigma_{N-1})^{\sigma}\ J_{n-1}'
$$

$$
\vdots
$$

$$
a : \Phi \qquad\qquad\qquad\qquad P
$$

where $a_n : \Gamma_n$ denotes the lowersequent of the trace $(\Sigma_{N-1})^{\sigma}\ J_{n-1}'$ of J_{n-1}, and $a : \Phi$ the bar of the rule $(c)_{\sigma}\ J_{n-1}$. Let $(\Sigma_{N-1})^{\sigma_{i+1}}\ J_i'$ denote the trace of J_i for $0 \leq i < n$. Put

$$
h := \mathrm{h}(a; P).
$$

By Lemma 5.2 there is no chain passing through the bar $a : \Phi$.

Let P' be the following:

$$
\dfrac{
\begin{array}{c}
\vdots\\
\Gamma, A\\
\vdots\\
\dfrac{a_0^l : \Gamma_0, A}{\Gamma_0', A^{\sigma_1}}\,J_0^l\\
\vdots\\
\dfrac{a_i^l : \Gamma_i, A^{\sigma_i}}{\Gamma_i', A^{\sigma_{i+1}}}\,J_i^l\\
\vdots\\
\dfrac{a_{n-1}^l : \Gamma_{n-1}, A^{\sigma_{n-1}}}{\Gamma_{n-1}', A^{\sigma}}\,J_{n-1}^l\\
\vdots\\
\dfrac{\overline{a_n^l : \Gamma_n, A^{\sigma}}}{\Gamma_n, A^{\rho}}\,(c\Pi_N)^{\sigma}_{\rho}\,J_n\\
\vdots\\
\Phi, A^{\rho}
\end{array}
\qquad
\begin{array}{c}
\vdots\\
\neg A^{\rho}, \Gamma\\
\vdots\\
\dfrac{a_0^r : \neg A^{\rho}, \Gamma_0}{\neg A^{\rho}, \Gamma_0'}\,J_0^r\\
\vdots\\
\dfrac{a_i^r : \neg A^{\rho}, \Gamma_i}{\neg A^{\rho}, \Gamma_i'}\,J_i^r\\
\vdots\\
\dfrac{a_{n-1}^r : \neg A^{\rho}, \Gamma_{n-1}}{\neg A^{\rho}, \Gamma_{n-1}'}\,J_{n-1}^r\\
\vdots\\
\dfrac{\overline{a_n^r : \neg A^{\rho}, \Gamma_n}}{\neg A^{\rho}, \Gamma_n}\,(w)\,J_n^r\\
\vdots\\
\neg A^{\rho}, \Phi
\end{array}
}{a : \Phi}\,(\Sigma_N)^{\rho}\,J_n' \qquad P'
$$

For the proviso **(lbranch)** in P', any ancestor of the left cut formula of the new $(\Sigma_N)^{\rho}\,J_n'$ is a genuine Π_N^{τ}-formula A^{τ} for a τ with $\rho \preceq \tau$. The formula A^{τ} is not in the branch $\mathcal{T}$ from $r : \Gamma$ to $a : \Phi$ in P since no genuine Π_N^{τ}-formula with $\tau > \Omega$ is on the rightmost branch $\mathcal{T}$. Therefore any left branch of the new $(\Sigma_N)^{\rho}\,J_n'$ is the rightmost one in the left upper part of the J_n' in P'.

In P', a new chain $J_0^l, \ldots, J_{n-1}^l, J_n$ starting with the new J_n is in the chain analysis for P' and $\rho = d_{\sigma}^{q}\alpha \in \mathcal{D}_{\sigma}$ is determined as follows:

$$b(\rho) = \alpha =$$

$$\max\{\mathcal{B}_{\pi}(o(a_n^l; P')), \mathcal{B}_{>\sigma}(\{\sigma\} \cup (a_n; P))\} + \omega^{o(a_n^l; P')} + \max\{\mathcal{K}_{\sigma}(a_n; P), \mathcal{K}_{\sigma}(h)\},$$

$rg_{N-1}(\rho) = \pi$ and $st_{N-1}(\rho) = o(a_0^l; P')$.

Let

$$0 \leq n_0 < n_1 < \cdots < n_l = n-1 \, (l \geq 0) \tag{11}$$

be the knotting numbers of the rope $\mathcal{R}$ and K_m an i_m-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}$ of J_{n_m} and J_{n_m+1} for $m < l$. Let $m(i)$ denote the number

$$m(i) = \max\{m : 0 \leq m \leq l \,\&\, \forall p \in [0, m)(i \leq i_p)\}. \tag{13}$$

Then $pd_i(\rho), In(\rho), rg_i(\rho), st_i(\rho)$ are determined so as to enjoy the provisos **(ch:Qpt)** and **(st:bound)**.

1. $pd_i(\rho) = \sigma_{n_{m(i)}+1}$ for $2 \leq i < N$. Note that $pd_i(\rho) \neq \pi = \sigma_0$ since $n_0 \geq 0$, cf. the condition (8) in Section 4 which says that $pd_{N-1}(\rho) = \pi \Leftrightarrow \sigma = \pi$.

2. $N-1 \in In(\rho)$ and $i \in In(\rho) \Leftrightarrow \exists p \in [0, m(i))(i_p = i)$ for $2 \leq i < N-1$.

3. Let $i \in In(\rho) \,\&\, i \neq N-1$. q denotes a number determined as follows.

   Case 1 The case when there exists a q such that

$$\exists p[n_{m(i)} \geq p \geq q > n_{m(i+1)} \,\&\, \rho \prec_i \sigma_{p+1} \,\&\, \sigma_q = rg_i(\sigma_{p+1})] \tag{14}$$

   Then q denotes the minimal q satisfying (14). Note that $\rho \prec_i \sigma_{p+1}$ is equivalent to $pd_i(\rho) = \sigma_{n_{m(i)}+1} \preceq_i \sigma_{p+1}$.

   Case 2 Otherwise. Then set $q = n_{m(i)} + 1$.

   In each case set $rg_i(\rho) = \sigma_q := \kappa$ for the number q, and $st_i(\rho) = d_{\kappa^+}\alpha_i$ for

$$\alpha_i = B_\kappa(a_q^l; P')$$

   where a_q^l denotes the uppersequent Γ_q, A^{σ_q} of J_q^l in the left upper part of $(\Sigma_N)^\rho\, J_n'$ in P'.

   By Lemma 4.2 we have $\mathcal{B}_{>\kappa^+}(\alpha_i) \subset \mathcal{B}_{>\kappa}(\alpha_i) < \alpha_i$, and hence $st_i(\rho) \in Od(\Pi_N)$.

Obviously the provisos **(ch:Qpt)** and **(st:bound)** are enjoyed for the new chain $J_0^l, \ldots, J_{n-1}^l, J_n$.

Observe that, cf. (9) in Section 4,

$$\pi < \beta \in q = Q(\rho) \Rightarrow \beta = st_{N-1}(\rho).$$

Claim 6.1 $\rho = d_\sigma^q \alpha \in Od(\Pi_N)$.

Proof of Claim 6.1.
(5) $\mathcal{B}_{>\sigma}(\{\sigma, \alpha\} \cup q) < \alpha$: By Lemma 4.2 we have $\mathcal{B}_{>\sigma}(\{\sigma, \alpha\}) < \alpha$. It suffices to see $\mathcal{B}_{>\sigma}(q) < \alpha$. By the definition we have $\{pd_i(\rho), rg_i(\rho) : i \in In(\rho)\} \subset \{\sigma_p, \sigma_p^+ : p \leq n\}$. On the other hand we have $\mathcal{B}_{>\sigma}(\{\sigma_p, \sigma_p^+ : p \leq n\}) \subset \mathcal{B}_{>\sigma}(\sigma)$.

We have $\mathcal{B}_{>\sigma}(st_{N-1}(\rho)) \subset \mathcal{B}_{>\sigma}(\alpha)$. Finally for $st_i(\rho) = d_{\kappa^+}\alpha_i$ with $i < N-1$, we have $\mathcal{B}_{>\sigma}(st_i(\rho)) \subset \mathcal{B}_{>\sigma}(\{\sigma, \alpha_i\}) \cup \{\alpha_i\}$, and $\mathcal{B}_{>\sigma}(\{\sigma, \alpha_i\}) \subset \mathcal{B}_{>\sigma}(\{\sigma, \alpha\})$ and $\alpha_i < \alpha$.

($\mathcal{D}^Q$.12):
Case 2 This corresponds to ($\mathcal{D}^Q$.12.1), $\kappa = rg_i(\rho) = pd_i(\rho)$. Let α_1 denote the diagram such that $\rho \preceq \alpha_1 \in \mathcal{D}_\kappa$. Then

$$\alpha_1 = \sigma_{n_{m(i)}+2}\, (pd_i(\rho) = \sigma_{n_{m(i)}+1} \,\&\, \sigma_{n+1} = \rho).$$

We have by Lemma 4.2 and **(c:bound2)**,

$$\mathcal{B}_{>\sigma}(B_\kappa(a_{n_{m(i)}+1}; P)) < B_\kappa(a_{n_{m(i)}+1}; P) \leq b(\alpha_1).$$

On the other hand we have for $st_i(\rho) = d_{\kappa^+}\alpha_i$

$$\mathcal{B}_{>\kappa}(st_i(\rho)) \subset \mathcal{B}_{>\kappa}(\{\kappa, \alpha_i\}) \leq \mathcal{B}_{>\sigma}(B_\kappa(a_{n_{m(i)}+1}; P)).$$

Thus $\mathcal{B}_{>\kappa}(st_i(\rho)) < b(\alpha_1)$.
Case 1 This corresponds to $(\mathcal{D}^Q.12.2)$, $rg_i(\rho) = rg_i(pd_i(\rho))$ or to $(\mathcal{D}^Q.12.3)$, $rg_i(pd_i(\rho)) \prec_i \kappa$ by Lemma 5.8.1c. Let p denote the maximal p such that

$$rg_i(\sigma_{p+1}) = \sigma_q = rg_i(\rho) \,\&\, pd_i(\rho) \preceq_i \sigma_{p+1}.$$

$st_i(\rho) < st_i(pd_i(\rho))$ for the case $(\mathcal{D}^Q.12.2)$ and $st_i(\rho) < st_i(\sigma_{p+1})$ for the case $(\mathcal{D}^Q.12.3)$ follow from Lemma 5.8.2 since for $rg_i(\sigma_{p+1}) = \sigma_q = rg_i(\rho)$

$$b(st_i(\rho)) = B_{\sigma_q}(a_q^l; P') < B_{\sigma_q}(a_q; P) \leq b(st_i(\sigma_{p+1}))$$

and hence by Lemmata 4.1 and 4.2

$$st_i(\rho) < st_i(\sigma_{p+1}).$$

$(\mathcal{D}^Q.11)$ and $(\mathcal{D}^Q.12.3)$: These follow from Lemma 5.8.1.
$(\mathcal{D}^Q.2)$: $\forall\tau \leq rg_i(\rho)(K_\tau st_i(\rho) < \rho)$. For $\tau \leq \kappa = rg_i(\rho)$ and $st_i(\rho) = d_{\kappa^+}\alpha_i$, we have $K_\tau(st_i(\rho)) = K_\tau\{\kappa, \alpha_i\} \leq \mathcal{K}_\tau(a_q^l; P') < \rho$ as in the case **M6.2** in [6]. □

As in the case **M6.2** in [6] we see that $o(P') < o(P)$.

We have to verify that P' is a proof. The provisos other than **(uplwl)** are seen to be satisfied as in the case **M5.2** of [6]. For the proviso **(forerun)** see Claim 6.3 in the subcase **M7.2** below. It suffices to see that P' enjoys the proviso **(uplwl)** when the lower rule J^{lw} is the new $(\Sigma_N)^\rho\, J'_n$. For example the left rope ${}_{K_m}\mathcal{R}$ of the i_m-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}\, K_m$ of J_{n_m} and J_{n_m+1} ends with the rule $(c)_\sigma\, J_{n-1}$. We show the following claim.

Claim 6.2 *Any left rope ${}_{J^{up}}\mathcal{R}$ of a knot J^{up} in the left upper part of the new $(\Sigma_N)^\rho\, J'_n$ does not reach to J'_n.*

Proof of Claim 6.2. Consider the original proof P. By Lemma 5.2 there is no chain passing through the bar $a : \Phi$ and hence it suffices to see that there is no rule $(c)_\rho^\sigma$ above $a : \Phi$. First observe that we have $\rho < \tau$ for any rules $(c)_\tau$ and $(\Sigma_i)^\tau$ which are between $(\Pi_N\text{-rfl})\, J$ and $a : \Phi$. Thus there is no rule $(c)_\rho^\sigma$ on the branch $\mathcal{T}_0$ from $(c)^\pi\, J_0$ to $a : \Phi$. Consider another branch $\mathcal{T}$ above $a : \Phi$ and suppose that there is a rule $(c)_\rho^\sigma\, I$ on $\mathcal{T}$. We can assume that the merging rule K of $\mathcal{T}$ and $\mathcal{T}_0$ is below J_0 and hence the rule K is a $(\Sigma_i)^\tau$. By the proviso **(h-reg)**(cf. Definition 5.4.4 in [6].) we have $\tau \leq \sigma$, i.e., K is between $(c)_\sigma^{\sigma_{n-1}}\, J_{n-1}$ and $a : \Phi$. Then we have seen $\rho < \tau$ and hence the trace $(\Sigma_{N-1})^\rho\, I_0$ of $(c)_\rho^\sigma\, I$ is below K by the proviso **(h-reg)**. Therefore the chain stating with the trace I_0

passes through the left side of K. This is impossible by the proviso **(ch:left)**.

$$
\begin{array}{c}
\begin{array}{cc}
\dfrac{\Psi_2}{\Psi'_2}\ (c)^{\sigma}_{\rho}\ I & \dfrac{\Gamma_{n-1}}{\Gamma'_{n-1}}\ (c)^{\sigma_{n-1}}_{\sigma}\ J_{n-1} \\
\vdots\ \mathcal{T} & \vdots\ \mathcal{T}_0
\end{array} \\
\dfrac{\Psi_1, \neg C^{\tau} \qquad C^{\tau}, \Phi_1}{\Psi_1, \Phi_1}\ (\Sigma_i)^{\tau}\ K \\
\vdots \qquad \vdots \\
\dfrac{\Psi_0, \neg B^{\rho} \quad B^{\rho}, \Phi_0}{\Psi_0, \Phi_0}\ (\Sigma_{N-1})^{\rho}\ I_0 \\
\vdots\ \mathcal{T}, \mathcal{T}_0 \\
a : \Phi
\end{array}
\qquad P
$$

In what follows we assume that $r * (0) = J$ is a basic rule. Let $v * (0) = I$ denote the vanishing cut of $r * (0) = J$. $v * (0) = I$ is either a (Σ_i) or a (cut).

M6. I is a $(\Sigma_N)^{\sigma}$.

$$
\dfrac{\begin{array}{cc} \vdots & \begin{array}{c} \vdots \qquad \vdots \\ \dfrac{\alpha < \sigma, \Lambda_0 \quad \neg A^{\sigma}_{N-1}(\alpha), \Lambda_0}{\exists x < \sigma \neg A^{\sigma}_{N-1}(x), \Lambda_0}\ (b\exists)\ J \\ \vdots \end{array} \\ \Gamma, A^{\sigma} & \neg A^{\sigma}, \Lambda \end{array}}{v : \Gamma, \Lambda}\ (\Sigma_N)^{\sigma}\ I \qquad P
$$

where $A \equiv \forall x A_{N-1}(x)$ is a Π_N formula.
Assuming $\alpha < \sigma$ let P' be the following:

$$
\dfrac{\dfrac{\begin{array}{cc} \vdots & \begin{array}{c} \vdots \\ \dfrac{\neg A^{\sigma}_{N-1}(\alpha), \Lambda_0}{\neg A^{\sigma}, \Lambda_0, \neg A^{\sigma}_{N-1}(\alpha)}\ (w) \\ \vdots \end{array} \\ \Gamma, A^{\sigma} & \neg A^{\sigma}, \Lambda, \neg A^{\sigma}_{N-1}(\alpha) \end{array}}{\Gamma, \Lambda, \neg A^{\sigma}_{N-1}(\alpha)} \qquad \begin{array}{c} \vdots \\ \Gamma, A^{\sigma}_{N-1}(\alpha) \end{array}}{v : \Gamma, \Lambda}\ (\Sigma_{N-1})^{\sigma}\ I_{N-1} \qquad P'
$$

where, the preproof ending with $\Gamma, A^{\sigma}_{N-1}(\alpha)$ is obtained from the left upper part of I in P by inversion.

As in the case **M6** of [6] we see that $o(v; P') < o(v; P)$.

For the proviso **(lbranch)** in P', cf. the case **M5.2**. We verify that P' is a proof with respect to the proviso **(uplw)**.

The proviso **(uplwl)** when the lower rule J^{lw} is the new $(\Sigma_{N-1})^\sigma\, I_{N-1}$: Consider the original proof P. By Lemma 5.3 no left rope in the right upper part of $(\Sigma_N)^\sigma\, I$ reaches to I. Also by **(uplwl)** with the lower rule $J^{lw} = I$ there is no left rope of an i-knot J^{up} reaching to I.

The proviso **(uplwr)** when the lower rule J^{lw} is the new $(\Sigma_{N-1})^\sigma\, I_{N-1}$: As above there is no left rope of an i-knot J^{up} reaching to I.

The proviso **(uplwr)** when the upper rule J^{up} is the $(\Sigma_{N-1})^\sigma\, I_{N-1}$: $(\Sigma_{N-1})^\sigma\, I_{N-1}$ is not an $(N-1)$-knot since there is no chain passing through $(\Sigma_N)^\sigma\, I$ by **(ch:pass)**.

For the proviso **(forerun)** see Claim 6.3 in the subcase **M7.2** below.

M7. I is a $(\Sigma_{i+1})^\sigma$ with $1 \leq i < N-1$.
Then J is either an $(\exists)$ or a $(b\exists)$. Let $u_0 : \Psi$ denote the uppermost sequent below I such that $h(u_0; P) < \sigma + i$. Also let $u : \Phi$ denote the resolvent of I, cf. Definition 5.5.
M7.1 $u_0 = u$.

$$\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Gamma, \neg A^\sigma_{i+1}\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \alpha < \tau, \Lambda_0\end{array} \quad \begin{array}{c}\vdots\\ A^\tau_i(\alpha), \Lambda_0\end{array}}{A^\tau_{i+1}, \Lambda_0}\, x \\ \vdots \\ A^\sigma_{i+1}, \Lambda \end{array}}{\Gamma, \Lambda}\,(\Sigma_{i+1})^\sigma\, I \\ \vdots \\ u : \Psi
\end{array} \qquad P$$

where $A_{i+1} \equiv \exists y A_i(y)$ is a Σ_{i+1} formula. Also if x is an $(\exists)$, then $\tau = \pi$ and the left upper part of the true sequent $\alpha < \tau, \Lambda_0$ is absent. Anyway $\sigma \preceq \tau$.

Assuming $\alpha < \tau$ and then $\alpha < \sigma$ by **(c:bound)**, let P' be the following:

$$\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Gamma, \neg A^\sigma_{i+1}\end{array} \quad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ A^\tau_i(\alpha), \Lambda_0\end{array}}{A^\tau_{i+1}, A^\tau_i(\alpha), \Lambda_0}\,(w) \\ \vdots \\ A^\sigma_{i+1}, A^\sigma_i(\alpha), \Lambda\end{array}}{A^\sigma_i(\alpha), \Gamma, \Lambda} \\ \vdots \\ \Psi, A^\sigma_i(\alpha)\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \Gamma, \neg A^\sigma_i(\alpha)\end{array}}{\Gamma, \Lambda, \neg A^\sigma_i(\alpha)}\,(w) \\ \vdots \\ \neg A^\sigma_i(\alpha), \Psi\end{array}}{u : \Psi}\,(\Sigma_i)^\sigma \qquad P'$$

It is easy to see that $o(u; P') < o(u; P)$. For the proviso **(lbranch)** in P', cf. the case **M5.2**. To see that P' is a proof with respect to the provisos **(forerun)**, **(uplw)**, cf. the subcase **M7.2** below.

M7.2 Otherwise.
Let K denote the lowermost rule $(\Sigma_{i+1})^{\sigma}$ below or equal to I. Then $u_0 : \Psi$ is the lowersequent of K by **(h-reg)**. There exists an $(i+1)$-knot $(\Sigma_{i+1})^{\sigma}$ which is between an uppersequent of I and $u_0 : \Psi$. Pick the uppermost such knot $(\Sigma_{j+1})^{\sigma}\, K_{-1}$ and let ${}_{K_{-1}}\mathcal{R} = J_0, \ldots, J_{n-1}$ denote the left rope of K_{-1}. Each J_p is a rule $(c)^{\sigma_p}_{\sigma_{p+1}}$ with $\sigma = \sigma_0$. Let

$$0 \leq n_0 < n_1 < \cdots < n_l = n-1 \, (l \geq 0) \tag{11}$$

be the knotting numbers of the left rope ${}_{K_{-1}}\mathcal{R}$ and K_m an i_m-knot $(\Sigma_{i_m})^{\sigma_{n_m+1}}$ of J_{n_m} and J_{n_m+1} for $m < l$. Put

$$m(i+1) = \max\{m : 0 \leq m \leq l \,\&\, \forall p \in [0, m)(i+1 \leq i_p)\} \tag{12}$$

Then the resolvent $u : \Phi$ is the uppermost sequent $u : \Phi$ below $J_{n_{m(i+1)}}$ such that

$$h(u; P) < \sigma_{n_{m(i+1)}+1} + i.$$

In the following figure of P the chain $\mathcal{C}_{n_{m+1}}$ starting with $J_{n_{m+1}}$ passes through

the left side of K_m.

$$
\dfrac{
\begin{array}{c} \vdots\ \mathcal{T}_l \\ \Gamma, \neg A^{\sigma}_{i+1} \end{array}
\qquad
\begin{array}{c} \dfrac{\alpha < \tau, \Lambda_0 \quad A^{\tau}_{i}(\alpha), \Lambda_0}{A^{\tau}_{i+1}, \Lambda_0}\ J \\ \vdots \\ A^{\sigma}_{i+1}, \Lambda \end{array}
}{v : \Gamma, \Lambda}\ (\Sigma^{\sigma}_{i+1})\, I
$$

$$
\begin{array}{c}
\vdots\ \mathcal{T}_1 \\
\dfrac{\Gamma_0}{\Gamma'_0}\ (c)^{\sigma}_{\sigma_1}\ J_0 \\
\vdots \\
\dfrac{\Gamma_{n_m}}{\Gamma'_{n_m}}\ (c)^{\sigma_{n_m}}_{\sigma_{n_m+1}}\ J_{n_m} \\
\vdots
\end{array}
$$

$$
\dfrac{
\begin{array}{c} \vdots\ \mathcal{C}_{n_{m+1}} \\ \Pi_m, \neg B_m \end{array}
\qquad
B_m, \Delta_m
}{\Pi_m, \Delta_m}\ (\Sigma_{i_m})^{\sigma_{n_m+1}}\ K_m
$$

$$
\begin{array}{c}
\vdots \\
\dfrac{\Gamma_{n_m+1}}{\Gamma'_{n_m+1}}\ (c)^{\sigma_{n_m+1}}_{\sigma_{n_m+2}}\ J_{n_m+1} \\
\vdots \\
\dfrac{\Gamma_{n_{m+1}}}{\Gamma'_{n_{m+1}}}\ (c)^{\sigma_{n_{m+1}}}_{\sigma_{n_{m+1}+1}}\ J_{n_{m+1}} \\
\vdots \\
\dfrac{\Gamma_{n_{m(i+1)}}}{\Gamma'_{n_{m(i+1)}}}\ (c)^{\sigma_{n_{m(i+1)}}}_{\sigma_{n_{m(i+1)}+1}}\ J_{n_{m(i+1)}} \\
\vdots\ \mathcal{T}_1 \\
u : \Phi
\end{array}
\qquad\qquad P
$$

Assuming $\alpha < \tau$ and then $\alpha < \sigma_n \leq \sigma_{n_{m(i)}+1}$, let P' be the following:

$$\dfrac{A_i^{\tau}(\alpha), \Lambda_0}{A_{i+1}^{\tau}, \Lambda_0, A_i^{\tau}(\alpha)}\ (w)$$

$$\dfrac{\begin{matrix}\vdots \\ \Gamma, \neg A_{i+1}^{\sigma}\end{matrix} \qquad \begin{matrix}\vdots \\ A_{i+1}^{\sigma}, \Lambda, A_i^{\sigma}(\alpha)\end{matrix}}{\Gamma, \Lambda, A_i^{\sigma}(\alpha)}\ I^l \qquad\qquad \dfrac{\begin{matrix}\vdots\ \mathcal{T}_r \\ \Gamma, \neg A_i^{\sigma}(\alpha)\end{matrix}}{v^r : \neg A_i^{\sigma}(\alpha), \Gamma, \Lambda}\ (w)$$

$$\vdots \qquad\qquad\qquad \vdots\ \mathcal{T}_1^r \subset \mathcal{T}_r$$

$$\dfrac{\Gamma_0, A_i^{\sigma}(\alpha)}{\Gamma'_0, A_i^{\sigma_1}(\alpha)}\ J_0^l \qquad\qquad \dfrac{\neg A_i^{\sigma}(\alpha), \Gamma_0}{\neg A_i^{\sigma_1}(\alpha), \Gamma'_0}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\dfrac{\Gamma_{n_m}, A_i^{\sigma_{n_m}}(\alpha)}{\Gamma'_{n_m}, A_i^{\sigma_{n_m+1}}(\alpha)}\ J_{n_m}^l \qquad\qquad \dfrac{\neg A_i^{\sigma_{n_m}}(\alpha), \Gamma_{n_m}}{\neg A_i^{\sigma_{n_m+1}}(\alpha), \Gamma'_{n_m}}$$

$$\dfrac{\begin{matrix}\vdots\ \mathcal{C}_{n_{m+1}}^l \\ \Pi_m, \neg B_m\end{matrix} \qquad \begin{matrix}\vdots \\ B_m, \Delta_m, A_i^{\sigma_{n_m+1}}(\alpha)\end{matrix}}{\Pi_m, \Delta_m, A_i^{\sigma_{n_m+1}}(\alpha)}\ K_m^l \qquad \dfrac{\begin{matrix}\vdots\ \mathcal{C}_{n_{m+1}}^r \\ \Pi_m, \neg B_m\end{matrix} \qquad \begin{matrix}\vdots \\ \neg A_i^{\sigma_{n_m+1}}(\alpha), B_m, \Delta_m\end{matrix}}{\neg A_i^{\sigma_{n_m+1}}(\alpha), \Pi_m, \Delta_m}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\dfrac{\Gamma_{n_m+1}, A_i^{\sigma_{n_m+1}}(\alpha)}{\Gamma'_{n_m+1}, A_i^{\sigma_{n_m+2}}(\alpha)}\ J_{n_m+1}^l \qquad\qquad \dfrac{\neg A_i^{\sigma_{n_m+1}}(\alpha), \Gamma_{n_m+1}}{\neg A_i^{\sigma_{n_m+2}}(\alpha), \Gamma'_{n_m+1}}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\dfrac{\Gamma_{n_{m+1}}, A_i^{\sigma_{n_{m+1}}}(\alpha)}{\Gamma'_{n_{m+1}}, A_i^{\sigma_{n_{m+1}+1}}(\alpha)}\ J_{n_{m+1}}^l \qquad\qquad \dfrac{\neg A_i^{\sigma_{n_{m+1}}}(\alpha), \Gamma_{n_{m+1}}}{\neg A_i^{\sigma_{n_{m+1}+1}}(\alpha), \Gamma'_{n_{m+1}}}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\dfrac{\Gamma_{n_{m(i+1)}}, A_i^{\sigma_{n_{m(i+1)}}}(\alpha)}{\Gamma'_{n_{m(i+1)}}, A_i^{\sigma_{n_{m(i+1)}+1}}(\alpha)}\ J_{n_{m(i+1)}}^l \qquad\qquad \dfrac{\neg A_i^{\sigma_{n_{m(i+1)}}}(\alpha), \Gamma_{n_{m(i+1)}}}{\neg A_i^{\sigma_{n_{m(i+1)}+1}}(\alpha), \Gamma'_{n_{m(i+1)}}}$$

$$\vdots \qquad\qquad\qquad \vdots\ \mathcal{T}_1^r \subset \mathcal{T}_r$$

$$\dfrac{\Phi, A_i^{\sigma_{n_{m(i+1)}+1}}(\alpha) \qquad\qquad u*(1) : \neg A_i^{\sigma_{n_{m(i+1)}+1}}(\alpha), \Phi}{u : \Phi}\ I_i$$

$$P'$$

Here I_i denotes a $(\Sigma_i)^{\sigma_{n_{m(i+1)}+1}}$.

It is straightforward to see $o(u; P') < o(u; P)$. We show P' is a proof.

First by Lemma 5.5, in P every chain passing through the resolvent $u : \Phi$ passes through the right side of I and, by inversion, the right upper part of I disappears in P'. Hence the new $(\Sigma_i)^{\sigma_{n_{m(i+1)}+1}}$ I_i does not split any chain. For the proviso **(lbranch)** in P', cf. the case **M5.2**.

Claim 6.3 *The proviso* **(forerun)** *holds for the lower rule $J^{lw} = I_i$ in P'.*

Proof of Claim 6.3. Consider a right branch $\mathcal{T}_r$ of I_i. We show that there is no rule K such that $\mathcal{T}_r$ passes through the left side of K and $h(a;P') < \pi$ with the lowersequent a of K. The assertion follows from this and **(h-reg)**. The ancestors of the right cut formula $\neg A_i^{\sigma_{n_{m(i+1)}+1}}(\alpha)$ of I_i comes from the left cut formula $\neg A^{\sigma}_{i+1}$ of I in P. Let $\mathcal{T}_1^r$ denote the branch in P' from the lowersequent $v^r : \neg A^{\sigma}_i(\alpha), \Gamma, \Lambda$ of the new (w) to the right uppersequent $u*(1)$: $\neg A_i^{\sigma_{n_{m(i+1)}+1}}(\alpha), \Phi$ of I_i. Also let $\mathcal{T}_l$ denote a (the) left branch of I in P. There exists a (possibly empty) branch $\mathcal{T}_0$ such that $\mathcal{T}_r = \mathcal{T}_0{}^\frown\mathcal{T}_l{}^\frown\mathcal{T}_1^r$. By **(lbranch)** any left branch $\mathcal{T}_l$ of I is the rightmost one in the left upper part of I. Therefore there is no rule K such that $\mathcal{T}_r$ passes through the left side of K and $h(a;P') < \pi$ with the lowersequent a of K. $\Box$

Claim 6.4 *The proviso* **(uplwr)** *holds for the upper rule $J^{up} = I_i$ in P'.*

Proof of Claim 6.4. Suppose that I_i is a knot. Then there exists a chain $\mathcal{C}_1$ starting with an I_1 such that $\mathcal{C}_1$ passes through the left side of I_i. This chain comes from a chain in P which passes through $u : \Phi$. Call the latter chain in P $\mathcal{C}_1$ again. Further assume that, in P', the left rope ${}_{I_i}\mathcal{R}$ of I_i reaches to a rule $(\Sigma_j)^{\kappa}\ J^{lw}$ with $i \leq j$. Let I_2 denote the lower rule of I_i. We have to show I_i foreruns J^{lw}. It suffices to show that, in P, any right branch $\mathcal{T}$ of J^{lw} passes through the right side of I if the branch $\mathcal{T}$ passes through $u : \Phi$. Since, by inversion, the right upper part of I disappears in P', for such a branch $\mathcal{T}$ there exists a unique branch $\mathcal{T}'$ corresponding to it in P' so that $\mathcal{T}'$ passes through the left side of I_i and hence $\mathcal{T}'$ is left to I_i.

$$
\dfrac{\begin{array}{c}\vdots\\ \Gamma_{lw}, \neg C_{lw}\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \Gamma, \neg A^{\sigma}_{i+1}\end{array} \quad \begin{array}{c}\vdots\ \mathcal{T}\\ A^{\sigma}_{i+1}, \Lambda\end{array}}{\Gamma, \Lambda}\ (\Sigma_{i+1})^{\sigma}\ I \\ \vdots \\ u : \Phi \\ \vdots\ \mathcal{T} \\ C_{lw}, \Lambda_{lw}\end{array}}{\Gamma_{lw}, \Lambda_{lw}}\ (\Sigma_j)^{\kappa}\ J^{lw} \qquad\qquad P
$$

$$
\dfrac{\begin{array}{c}\vdots\\ \Gamma_{lw}, \neg C_{lw}\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \Gamma, \neg A^{\sigma}_{i+1}\end{array} \quad \begin{array}{c}\vdots\ \mathcal{T}'\\ A^{\sigma}_{i+1}, \Lambda, A^{\sigma}_i(\alpha)\end{array}}{\Gamma, \Lambda, A^{\sigma}_i(\alpha)}\ I^l \\ \vdots \\ \Phi, A_i^{\sigma_{n_{m(i)}+1}}(\alpha)\end{array} \qquad \begin{array}{c}\vdots\\ \neg A_i^{\sigma_{n_{m(i)}+1}}(\alpha), \Phi\end{array}}{u : \Phi}\ I_i \\ \vdots\ \mathcal{T}' \\ C_{lw}, \Lambda_{lw}\end{array}}{\Gamma_{lw}, \Lambda_{lw}}\ J^{lw} \qquad\qquad P'
$$

Case 1. The case when, in P, there exists a member I_3 of the chain $\mathcal{C}_1$ such that I_3 is between $u : \Phi$ and J^{lw}, and the chain $\mathcal{C}_3$ starting with I_3 passes through the resolvent $u : \Phi$ in P: Then by Lemma 5.5 the chain $\mathcal{C}_3$ passes through the right side of I. The rope $\mathcal{R}_{I_3}$ starting with I_3 in P corresponds to a part (a tail) of the left rope ${}_{I_i}\mathcal{R}$ in P'. Thus by the assumption the rope $\mathcal{R}_{I_3}$ also reaches to J^{lw} in P. Hence by **(forerun)** there is no merging rule K such that

1. the chain $\mathcal{C}_3$ starting with I_3 passes through the right side of K, and
2. the right branch $\mathcal{T}$ of J^{lw} passes through the left side of K.

Therefore the right branch $\mathcal{T}$ of J^{lw} passes through the right side of I in P.

$$
\begin{array}{c}
\dfrac{\begin{array}{cc} \vdots & \vdots\ \mathcal{C}_3,\ \mathcal{T}' \\ \Gamma, \neg A^{\sigma}_{i+1} & A^{\sigma}_{i+1}, \Lambda, A^{\sigma}_i(\alpha) \end{array}}{\Gamma, \Lambda, A^{\sigma}_i(\alpha)}\ I^l \\
\dfrac{\begin{array}{cc} \vdots & \vdots \\ \Phi, A_i^{\sigma_{n_{m(i)}+1}}(\alpha) & \neg A_i^{\sigma_{n_{m(i)}+1}}(\alpha), \Phi \end{array}}{u : \Phi}\ I_i \\
\vdots \\
\dfrac{\Delta_2}{\Delta'_2}\ I_2 \\
\vdots \\
\dfrac{\Delta_3}{\Delta'_3}\ I_3 \\
\dfrac{\begin{array}{cc} \vdots & \vdots\ \mathcal{R}_{I_3} \subset {}_{I_i}\mathcal{R} \\ \Gamma_{lw}, \neg C_{lw} & C_{lw}, \Lambda_{lw} \end{array}}{\Gamma_{lw}, \Lambda_{lw}}\ J^{lw} \qquad\qquad P'
\end{array}
$$

Case 2. Otherwise: First we show the following claim:

Claim 6.5 *In P, we have $m(i+1) < l$ for the number of knots l in (11), and I_2 is the lower rule of the $i_{m(i+1)}$-knot $K_{m(i+1)}$. Let ${}_{K_{m(i+1)}}\mathcal{R}$ denote the left rope of $K_{m(i+1)}$ in P. Then ${}_{K_{m(i+1)}}\mathcal{R}$ reaches to J^{lw}.*

Proof of Claim 6.5. In P', the lower rule I_2 of the knot I_i is a member of the chain $\mathcal{C}_1$ starting with I_1 and passing through the left side of I_i. Further I_2 is above J^{lw} since the left rope ${}_{I_i}\mathcal{R}$ of I_i is assumed to reach to J^{lw} in P', cf. Definition 5.7. Since we are considering when **Case 1** is not the case, in P, I_1 is below J^{lw} and the chain $\mathcal{C}_2$ starting with I_2 does not pass through $u : \Phi$, and hence chains $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect as **Type3 (merge)** in **(ch:link)**. In other words there is a knot below $u : \Phi$ whose upper right rule is $(c)_{\sigma_{n_{m(i+1)}+1}}\, J_{n_{m(i+1)}}$. This means that the knot is the $i_{m(i+1)}$-knot $K_{m(i+1)}$. Thus we have shown that $m(i+1) < l$ and I_2 is the lower rule of the $i_{m(i+1)}$-knot $K_{m(i+1)}$.

Next we show that the left rope ${}_{K_{m(i+1)}}\mathcal{R}$ of $K_{m(i+1)}$ reaches to J^{lw} in P. Suppose this is not the case. Let $(c)_{\kappa_4}\, I_4$ denote the lowest (last) member of the

left rope ${}_{K_{m(i+1)}}\mathcal{R}$. Then $\kappa < \kappa_4$ for the rule $(\Sigma_j)^{\kappa}\, J^{lw}$. By $\kappa < \kappa_4$, the next member $(c)^{\kappa_4}\, I_5$ of the chain $\mathcal{C}_1$ is above J^{lw}. Since we are considering when **Case 1** is not the case, the chain $\mathcal{C}_5$ starting with I_5 does not pass through $u : \Phi$. By Definition 5.4.6 of left ropes and **(ch:link)** there would be a knot K' whose lower rule is I_5 and whose upper right rule is I_4. This is a contradiction since I_4 is assumed to be the last member of the left rope ${}_{K_{m(i+1)}}\mathcal{R}$. This shows Claim 6.5.

In the following figure note that $u : \Phi$ is above $K_{m(i+1)}$ by **(h-reg)** and the definition of the resolvent $u : \Phi$.

$$
\dfrac{\dfrac{\vdots\ \mathcal{C}_2 = \mathcal{C}_{n_{m(i+1)+1}}}{\Pi_{m(i+1)}, \neg B_{m(i+1)}} \qquad \dfrac{\dfrac{\vdots\ \mathcal{C}_1}{u : \Phi}\ \vdots}{B_{m(i+1)}, \Delta_{m(i+1)}}}{\Pi_{m(i+1)}, \Delta_{m(i+1)}}\ (\Sigma_{i_{m(i+1)}})^{\sigma_{n_{m(i+1)+1}}}\ K_{m(i+1)}
$$

$$\vdots$$

$$\frac{\Delta_2}{\Delta_2'}\ (c)^{\sigma_{n_{m(i+1)+1}}}\ I_2$$

$$\vdots$$

$$\frac{\Delta_4}{\Delta_4'}\ (c)_{\kappa_4}\ I_4$$

$$\frac{\vdots\ \mathcal{C}_5 \qquad \vdots}{\dfrac{\Pi, \neg B \qquad B, \Delta}{\Pi, \Delta}\ K'}$$

$$\vdots$$

$$\frac{\Delta_5}{\Delta_5'}\ (c)^{\kappa_4}\ I_5$$

$$\frac{\vdots \qquad\qquad \vdots}{\dfrac{\Gamma_{lw}, \neg C_{lw} \quad C_{lw}, \Lambda_{lw}}{\Gamma_{lw}, \Lambda_{lw}}\ (\Sigma_j)^{\kappa}\ J^{lw}}$$

$$\vdots\ \mathcal{C}_1$$

$$\frac{\Delta_1}{\Delta_1'}\ I_1$$

P

$\square$

By Claim 6.5, **(uplwr)** and $i_{m(i+1)} \leq i \leq j$, $K_{m(i+1)}$ foreruns J^{lw} in P. Therefore the right branch $\mathcal{T}$ of J^{lw} is left to $K_{m(i+1)}$. Also by **(h-reg)** $K_{m(i+1)}$ is below $u : \Phi$. Hence $\mathcal{T}$ does not pass through $u : \Phi$ in this case. This shows Claim 6.4. In the following figure $\mathcal{C}_2$ denotes the chain starting with I_2.

$$
\dfrac{\begin{array}{c}\vdots\\ \Gamma_{lw}, \neg C_{lw}\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ \mathcal{C}_2\\ \Pi_{m(i+1)}, \neg B_{m(i+1)}\end{array}
\quad
\begin{array}{c}u : \Phi\\ \vdots\\ B_{m(i+1)}, \Delta_{m(i+1)}\end{array}}{\Pi_{m(i+1)}, \Delta_{m(i+1)}}\ (\Sigma_{i_{m(i+1)}})^{\sigma_{n_{m(i+1)}}+1}\ K_{m(i+1)}\\
\vdots\\
\dfrac{\Delta_2}{\Delta'_2}\ I_2\\
\vdots\ {}_{K_{m(i+1)}}\mathcal{R}\\
C_{lw}, \Lambda_{lw}
\end{array}}{\Gamma_{lw}, \Lambda_{lw}}\ J^{lw}
$$

P

$\square$

Claim 6.6 *The proviso* **(uplw)** *holds for the lower rule $J^{lw} = I_i$ in P'.*

Proof of Claim 6.6. Let J^{up} be a j-knot (Σ_j) above I_i. Let H_0 denote the lower rule of J^{up}. Assume that the left rope ${}_{J^{up}}\mathcal{R} = H_0, \ldots, H_{k-1}$ of J^{up} reaches to the rule I_i. We show

$$i < j$$

even if J^{up} is in the right upper part of I_i. Consider the corresponding rule J^{up} in P.

Case 1 Either J^{up} is I or between I and $u : \Phi$: If either J^{up} is I or an i_m-knot K_m with $m < m(i+1)$, then $i < i+1 = j$ or $i < i_m = j$ by (12), resp.

Otherwise J^{up} is between K_{m-1} and J_{n_m} for some m with $0 \leq m \leq m(i+1)$. Then the rule J^{up} is the merging rule of the chain $\mathcal{C}_{n_m}$ starting with J_{n_m} and the chain $\mathcal{C}_{H_0}$ starting with H_0 so that $\mathcal{C}_{n_m}$ passes through the right side of J^{up} and $\mathcal{C}_{H_0}$ the left side of J^{up}. Hence by **(ch:link) Type3 (merge)** the rule H_{k-1} is above J_{n_m} and the left rope ${}_{H_0}\mathcal{R}$ does not reach to I_i. Thus this is not the case.

$$
\begin{array}{c}
\begin{array}{c} \vdots\ \mathcal{C}_{n_m} \\ \dfrac{\Pi_{m-1}, \neg B_{m-1} \quad B_{m-1}, \Delta_{m-1}}{\Pi_{m-1}, \Delta_{m-1}}\ K_{m-1} \end{array} \\
\begin{array}{cc} \vdots\ \mathcal{C}_{H_q} & \vdots\ \mathcal{C}_{n_m} \end{array} \\
\dfrac{\Delta, \neg C \qquad C, \Psi}{\Delta, \Psi}\ J^{up} \\
\vdots \\
\dfrac{\Lambda_q}{\Lambda'_q}\ H_q \\
\vdots \\
\dfrac{\Lambda_{k-1}}{\Lambda'_{k-1}}\ H_{k-1} \\
\vdots \\
\dfrac{\Gamma_{n_m}}{\Gamma'_{n_m}}\ J_{n_m}
\end{array}
$$

where H_q denotes the lowermost member of ${}_{H_0}\mathcal{R}$ such that the chain $\mathcal{C}_{H_q}$ starting with H_q passes through the left side of J^{up}. By **(ch:link) Type3 (merge)** the rule H_q is above J_{n_m} and so on.

Case 2 J^{up} is in the right upper part of I: Then the left rope ${}_{H_0}\mathcal{R}$ reaches to I. Hence by Lemma 5.4, i.e., by **(uplwr)** we have $i < i+1 < j$.

Case 3 J^{up} is in the left upper part of I: Then the left rope ${}_{H_0}\mathcal{R}$ reaches to I. Hence by **(uplwl)** we have $i < i+1 < j$.

Case 4 Otherwise: Then there exists a rule K such that J^{up} is in the left upper part of K and K is between I and Φ. By **(h-reg), (ch:pass)** K is a rule $(\Sigma_p)^\kappa$. The left rope ${}_{H_0}\mathcal{R} = H_0, \ldots, H_{k-1}$ reaches to K. Hence by **(uplwl)** we have

$$p < j \tag{32}$$

$$
\dfrac{\dfrac{\dfrac{\vdots \qquad \vdots}{\Delta, \neg C \quad C, \Psi}}{\Delta, \Psi}\ J^{up} \quad \begin{array}{c}\vdots\\ \Gamma_K, \neg D\end{array} \qquad \dfrac{\dfrac{\vdots \qquad \vdots}{\Gamma, \neg A^\sigma_{i+1} \quad A^\sigma_{i+1}, \Lambda}}{\Gamma, \Lambda}\ (\Sigma^\sigma_{i+1})\ I \quad \begin{array}{c}\vdots\\ D, \Lambda_K\end{array}}{\begin{array}{c}\Gamma_K, \Lambda_K \\ \vdots \\ \Phi\end{array}}\ (\Sigma_p)^\kappa\ K
$$

Case 4.1 H_{k-1} is below K: Let K' denote the uppermost knot such that K' is equal to or below K, and there exists a member of ${}_{H_0}\mathcal{R}$ such that the chain

starting with the member passes through the left side of K'. Let H_q be the lowermost member of ${}_{H_0}\mathcal{R}$ such that the chain $\mathcal{C}_{H_q}$ starting with H_q passes through the left side of K'. If there exists a member of ${}_{H_0}\mathcal{R}$ such that the chain starting with the member passes through the left side of K, then K' is equal to K.

$$
\begin{array}{c}
\dfrac{\begin{array}{c} J^{up} \\ \vdots\, \mathcal{C}_{H_q} \\ \Gamma_K, \neg D \end{array} \qquad \begin{array}{c} \vdots \\ D, \Lambda_K \end{array}}{\Gamma_K, \Lambda_K}\ (\Sigma_p)^\kappa\ K = K' \\
\vdots \\
\dfrac{\Delta_q}{\Delta'_q}\ H_q
\end{array}
$$

Otherwise K' is below K and it is a knot for the left rope ${}_{H_0}\mathcal{R}$. Let H_{q-1} denote the lowermost member of ${}_{H_0}\mathcal{R}$ above K. Then H_{q-1} is an upper right rule of the knot K' and K' is a rule $(\Sigma_{p'})^\kappa$ with

$$p' \leq p \tag{33}$$

by **(h-reg)**.

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c} \begin{array}{c} J^{up} \\ \vdots \\ \dfrac{\Delta_{q-1}}{\Delta'_{q-1}}\ H_{q-1} \\ \vdots \end{array} \\ \Gamma_K, \neg D \end{array} \qquad \begin{array}{c} I \\ \vdots \\ D, \Lambda_K \end{array}}{\Gamma_K, \Lambda_K}\ (\Sigma_p)^\kappa\ K}{} \\
\dfrac{\begin{array}{c} \vdots\, \mathcal{C}_{H_q} \\ \Gamma_{K'}, \neg D' \end{array} \qquad \begin{array}{c} \vdots \\ D', \Lambda_{K'} \end{array}}{\Gamma_{K'}, \Lambda_{K'}}\ (\Sigma_{p'})^\kappa\ K' \\
\vdots \\
\dfrac{\Delta_q}{\Delta'_q}\ H_q
\end{array}
$$

By Lemma 5.1 the uppermost member of $\mathcal{C}_{H_q}$ below K' is the lower rule of the knot K'. By (32), (33) and **Case 1** it suffices to show that the left rope ${}_{K'}\mathcal{R} = G_0, \ldots, G_{k_0}$ of K' reaches to I_i, i.e., to show the last member G_{k_0} is equal to or below the rule H_{k-1}. Then we will have $i < p' \leq p < j$.

Let $G_0 = H_{q_0}$ ($q_0 \geq q$) denote the lower rule of K' and G_{k_1} the lowermost member of ${}_{K'}\mathcal{R}$ such that the chain $\mathcal{C}_{G_{k_1}}$ starting with G_{k_1} passes through the left side of K'. Then by **(ch:link)** G_{k_1} is equal to or below H_q.

Case 4.1.1 $G_{k_1} = H_q$: Then $G_{k_0} = H_{k-1}$, i.e., $G_{q_1 - q_0} = H_{q_1}$ for any q_1 with

$q_0 \leq q_1 < k$.

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ \mathcal{C}_{H_q} \\ \Gamma_{K'}, \neg D'\end{array} \qquad \begin{array}{c}\vdots \\ D', \Lambda_{K'}\end{array}}{\Gamma_{K'}, \Lambda_{K'}}\ K' \\
\vdots \\
\dfrac{\Lambda_{q_0}}{\Lambda'_{q_0}}\ H_{q_0} = G_0 \\
\vdots \\
\dfrac{\Lambda_q}{\Lambda'_q}\ H_q = G_{k_1} \\
\dfrac{\begin{array}{c}\vdots\ \mathcal{C}_{H_{q_1}} \\ \Gamma_{K_1}, \neg D_1\end{array} \qquad \begin{array}{c}\vdots \\ D_1, \Lambda_{K_1}\end{array}}{\Gamma_{K_1}, \Lambda_{K_1}}\ K_1 \\
\vdots \\
\dfrac{\Lambda_{q+1}}{\Lambda'_{q+1}}\ H_{q+1} = G_{q+1-q_0} \\
\vdots \\
\dfrac{\Lambda_{q_1}}{\Lambda'_{q_1}}\ H_{q_1} = G_{q_1-q_0}
\end{array}
$$

where K_1 is a knot of $H_{q+1} = G_{q+1-q_0}$ and $H_q = G_{k_1}$ with $q+1-q_0 = k_1+1$.

Case 4.1.2 Otherwise: Then by **(ch:link)** G_{k_1} is already below H_{k-1}.

$$
\begin{array}{c}
\vdots\, \mathcal{C}_{H_q} \qquad \vdots\, \mathcal{C}_{G_{k_1}} \\
\dfrac{\Gamma_{mg}, \neg D_{mg} \quad D_{mg}, \Lambda_{mg}}{\Gamma_{mg}, \Lambda_{mg}}\ K_{mg} \\
\vdots \\
\dfrac{\Gamma_{K'}, \neg D' \quad D', \Lambda_{K'}}{\Gamma_{K'}, \Lambda_{K'}}\ K' \\
\vdots \\
\dfrac{\Lambda_{q_0}}{\Lambda'_{q_0}}\ H_{q_0} = G_0 \\
\vdots \\
\dfrac{\Lambda_q}{\Lambda'_q}\ H_q = G_{q-q_0} \\
\vdots\, \mathcal{C}_{H_{q_1}} \qquad \vdots \\
\dfrac{\Gamma_{K_1}, \neg D_1 \quad D_1, \Lambda_{K_1}}{\Gamma_{K_1}, \Lambda_{K_1}}\ K_1 \\
\vdots \\
\dfrac{\Lambda_{q+1}}{\Lambda'_{q+1}}\ H_{q+1} = G_{q+1-q_0} \\
\vdots \\
\dfrac{\Lambda_{q_1}}{\Lambda'_{q_1}}\ H_{q_1} = G_{q_1-q_0} \\
\vdots \\
\dfrac{\Lambda_{k-1}}{\Lambda'_{k-1}}\ H_{k-1} = G_{k-1-q_0} \\
\vdots \\
\dfrac{\Lambda_{k_1+q_0}}{\Lambda'_{k_1+q_0}}\ G_{k_1}
\end{array}
$$

where K_{mg} is a merging rule of the chain $\mathcal{C}_{H_q}$ starting with H_q and the chain $\mathcal{C}_{G_{k_1}}$ starting with G_{k_1}. Since the chain $\mathcal{C}_{H_{q+1}}$ starting with the lower rule $H_{q+1} = G_{q+1-q_0}$ of K_1 passes through the left side of K_1, G_{k_1} is not equal to H_{q+1} and hence is below H_{q+1} and so on.

Case 4.2 H_{k-1} is above K: Then H_{k-1} is a rule $(c)_{\sigma_{n_{m(i+1)}+1}}$ and K is a rule $(\Sigma_p)^{\sigma_{n_{m(i+1)}+1}}$. Let $d : \Gamma_K, \neg D$ denote an uppersequent of K. By **(h-reg)** and the definition of the sequent $u : \Phi$ we have $\sigma_{n_{m(i+1)}+1} + i \leq h(d; P) \leq \sigma_{n_{m(i+1)}+1} + p - 1$. Thus by (32) we get $i \leq p - 1 < j$.

$$
\begin{array}{ccc}
J^{up} & & \vdots \\
\vdots & & \\
\dfrac{\Lambda_{k-1}}{\Lambda'_{k-1}}\ (c)_{\sigma_{n_{m(i+1)}+1}}\ H_{k-1} & & \dfrac{\Gamma_{n_{m(i+1)}}}{\Gamma'_{n_{m(i+1)}}}\ (c)^{\sigma_{n_{m(i+1)}}}_{\sigma_{n_{m(i+1)}+1}}\ J_{n_{m(i+1)}} \\
\vdots & & \vdots
\end{array}
$$

$$
\dfrac{d : \Gamma_K, \neg D \qquad\qquad D, \Lambda_K}{\Gamma_K, \Lambda_K}\ (\Sigma_p)^{\sigma_{n_{m(i+1)}+1}}\ K
$$

$$
\begin{array}{c}
\vdots \\
\Phi \\
\vdots
\end{array}
$$

$$
\dfrac{\Pi_{m(i+1)}, \neg B_{m(i+1)} \qquad B_{m(i+1)}, \Delta_{m(i+1)}}{\Pi_{m(i+1)}, \Delta_{m(i+1)}}\ (\Sigma_{i_{m(i+1)}})^{\sigma_{n_{m(i+1)}+1}}\ K_{m(i+1)}
$$

where the $i_{m(i+1)}$-knot $K_{m(i+1)}$ disappears when $m(i+1) = l$ in (12).

This shows Claim 6.6. □

M8. I is a $(\Sigma_1)^\sigma$.
This is treated as in the case **M8** of [6].

Other cases are easy.

This completes a proof of the Main Lemma 5.1.

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